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Uncoupling of Damped Linear Potential Multi-Degrees-of-Freedom Structural and Mechanical Systems

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This paper provides the necessary and sufficient conditions for a multi-degrees-of-freedom linear potential system with an arbitrary damping matrix to be uncoupled into independent subsystems of at most two degrees-of-freedom using a real orthogonal transformation. The incorporation of additional information about the matrices, which many structural and mechanical systems commonly possess, shows a reduction in the number of these conditions to three. Several new results are obtained and are illustrated through examples. A useful general form for the damping matrix is developed that guarantees the uncoupling of such systems when they satisfy just two conditions. The results provided herein lead to new physical insights into the dynamical behavior of potential systems with general damping matrices and robust computational procedures. It is shown that the dynamics of a damped potential system in which the damping matrix may be arbitrary is identical to that of a damped gyroscopic potential system with a symmetric damping matrix. This brings, for the first time, these two systems, which are seen today as belonging to different categories of dynamical systems, under a unified framework. [DOI: 10.1115/1.4065568]

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1 Introduction

One of the major contributions of the Bernoullis (Johann and Daniel, eighteenth century) and Euler (1707–1783) to our understanding of the behavior of multi-degrees-of-freedom (MDOF) structural and mechanical potential systems is the determination of their natural frequencies of vibration, the existence of normal modes of vibration, and the idea of their superposition [1]. The extension to using normal mode analysis for linear damped MDOF potential systems by Caughey and O'Kelly was a major improvement in our understanding of systems that commonly arise in many engineering applications in aerospace, civil, and mechanical engineering, as well as in nature. At the heart of this improvement is an important theorem in linear algebra, namely, that two real symmetric matrices can be simultaneously diagonalized by a real orthogonal transformation if and only if they commute. Applying this primal theorem to a symmetric stiffness (potential) matrix and a symmetric damping matrix, often used to model a mechanical/structural system, enabled them to describe the oscillations of a damped MDOF potential system in terms of its classical normal modes [2]. The use of an orthogonal transformation to uncouple the dynamics of such a system is as simple and remarkable as it is useful since it provides: (1) a much-improved physical understanding of its vibratory behavior, and (2) robust computational methods to quantitatively determine it.

Today, symmetric damping matrices that commute with a symmetric stiffness matrix have come to be used commonly in the modeling of damped structural and mechanical MDOF systems. This is largely because such a representation of an MDOF linear dynamical system affords great simplicity in the analysis of its damped vibratory behavior, since it leads to the uncoupling of the system into smaller independent subsystems, each having just a single-degree-of-freedom. But, what if the damping matrix is not symmetric and/or does not commute with the stiffness matrix?

To the best of the authors' knowledge, little progress has been made in the use of real orthogonal transformations to uncouple potential systems that have more general (arbitrary) damping matrices and that do not satisfy the restrictions of symmetry and commutation. And yet, experimental results often yield damping matrices that may not be symmetric (and/or may not satisfy the needed commutation property) since the various damping sources (and their linearized approximations made during modeling) that can arise in complex structural and mechanical systems, such as spacecraft

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and building structures, can make the damping matrix lose the properties needed for classical normal modes to exist. Also, linear MDOF structural and mechanical systems that are controlled by using velocity feedback can have non-symmetric damping matrices. Despite the generally perceived need, from a practical science/engineering viewpoint, for an analogous (more general) theory that addresses linear MDOF potential systems that have more general damping matrices—a need that was well-recognized by Caughey [3]—the reason for its lack over the last 60 years or so can be traced back to the lack of primal results in linear algebra to base such a theory upon. In what follows we shall see that just as the advance provided by Caughey and O'Kelly was based on a primal theorem in linear algebra, such a result needs to be established in the context of general damping matrices; accordingly, our work, therefore, begins with two central results that we prove in linear algebra.

An approach for the complete diagonalization of linear MDOF systems with non-symmetric damping matrices using *complex equivalence* transformations (or biorthogonality) was investigated in Ref. [4]. As we shall illustrate, besides the use of complex vector spaces the transformation of damped MDOF potential systems to *diagonal* form through complex equivalence transformations is quite restrictive. We show (see *Remarks 5* and *6*) that while commonly occurring engineered systems as well as those found in nature can be decoupled to a quasi-diagonal form (a term explained later, after *Theorem 2* below) under suitable conditions, they often cannot be diagonalized using complex equivalence transformations.

In this paper, we study linear MDOF systems whose mass matrices are positive definite, whose stiffness matrices are symmetric, and whose damping matrices are arbitrary. We explore the conditions needed for a damping matrix to permit a linear MDOF system to be uncoupled through the use of a *real orthogonal transformation* and a simple *real* coordinate change in exactly the same manner done by Caughey and O'Kelly in Ref. [2]. We obtain the necessary and sufficient (n&s) conditions under which such MDOF systems can be uncoupled into smaller-dimensioned, independent subsystems, each with either a single-degree-of-freedom or two degrees-of-freedom (2DOF). In this sense, this paper represents an extension of the line of thinking first pioneered in Ref. [2]; it is an expansion of the idea of using *real orthogonal* transformations and *real* vector spaces to improve our physical understanding of general damped linear MDOF potential systems as well as to provide robust computational methods for their analysis. Such systems in which the damping matrix may not be symmetric and/or may not have the restrictive commutation properties required to yield classical normal modes often occur in real-life situations, and therefore the results obtained herein are expected to be of some practical value.

As an example of a system with a non-symmetric damping matrix, consider the linear two degrees-of-freedom system shown in Fig. 1 in which the mass m moves in a vertical x-y plane with

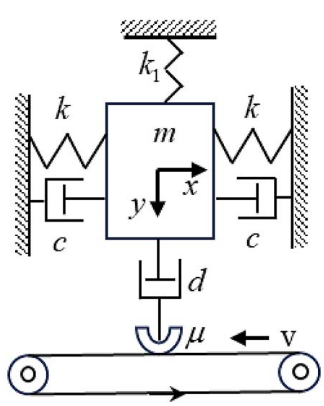

Fig. 1 Two degrees-of-freedom system with non-symmetric damping matrix

gravity acting downward. Motion in the y-direction is restrained by a linear spring with stiffness k_1, a linear dashpot with linear damping d, and a u-shaped snubber of mass m_1 that is in contact with a (straight) horizontal moving belt that has constant velocity v, as shown. Motion in the x-direction is restrained by linear springs each with spring constant k, dashpots each with linear damping c, and the frictional force between the snubber and the moving belt. The coefficient of kinetic friction between the snubber and the belt is μ so that the horizontal force on the snubber is $-\mu(d\dot{y}(t)+m_1 g)\mathrm{sgn}(v+\dot{x}(t))$ where g is the acceleration due to gravity. We assume that the belt velocity v is large enough so that $v+\dot{x}(t)>0$. The equation of motion for small oscillations about the equilibrium position is then given by

$$\underbrace{\begin{bmatrix} m+m_1 & 0 \\ 0 & m \end{bmatrix}}_{\tilde{M}}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}+\underbrace{\begin{bmatrix} 2c & \mu d \\ 0 & d \end{bmatrix}}_{\tilde{D}}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}+\underbrace{\begin{bmatrix} 2k & 0 \\ 0 & k_1 \end{bmatrix}}_{\tilde{K}}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We note that the matrices $\tilde{K}$ and $\tilde{D}$ do not commute when $k_1 \neq 2k$. The structure of the matrices in this simple example is meant to provide an easy conceptual notion for what we now take up in a more general setting.

Consider the general linear MDOF system described by the equation

$$\tilde{M}\ddot{q}+\tilde{D}\dot{q}+\tilde{K}q=\tilde{f}(t) \tag{1}$$

where $q(t)$ and $\tilde{f}(t)$ are n-vectors (n by 1 column vectors), $\tilde{M}^T=\tilde{M}>0$, $\tilde{K}=\tilde{K}^T$, and $\tilde{D}$ is an arbitrary damping matrix. The real matrices $\tilde{M}$, $\tilde{D}$, and $\tilde{K}$ are each n by n matrices, and the dots indicate differentiation with respect to time, t. Using the real transformation $q(t)=\tilde{M}^{-1/2}x(t)$, Eq. (1) reduces to the relation

$$\ddot{x}+D\dot{x}+Kx=f(t) \tag{2}$$

where

$$D=\tilde{M}^{-1/2}\tilde{D}\tilde{M}^{-1/2} \tag{3}$$

$$K=\tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2} \tag{4}$$

and

$$f(t)=\tilde{M}^{-1/2}\tilde{f}(t) \tag{5}$$

We shall refer to the matrices D and K as the damping and stiffness matrices, respectively, and $f(t)$ as the force. In what follows the matrix K is an n by n symmetric matrix while the elements of the n by n matrix $D \neq 0$ (unless otherwise specified) are arbitrary.

Lemma 1. *The real matrix* D *in Eq. (2) can be (uniquely) split into the sum of two matrices one of which is symmetric, the other skew-symmetric.*

Proof. (see, e.g., Horn and Johnson [5]) The matrix D can be uniquely split as

$$D=\frac{D+D^T}{2}+\frac{D-D^T}{2}:=S+G \tag{6}$$

where we have denoted the symmetric part of D by S and the skew-symmetric part of D by G. ∎

Using Eq. (6), Eq. (2) can be written as

$$\ddot{x}+\underbrace{(S+G)}_{D}\dot{x}+Kx=f(t) \tag{7}$$

Equation (7) is equivalent to Eq. (1) and we will be primarily using it in what follows. Furthermore, the n by n matrices S and K are always taken to be symmetric unless otherwise stated, and the n by n skew-symmetric matrix $G \neq 0$ is assumed to have $\text{Rank}(G) = 2m \leq n$ throughout the paper. The matrices S and G always refer to the symmetric and skew-symmetric parts of the n by n arbitrary damping matrix D shown in Eq. (6).

It should be noted that Eq. (7) can be interpreted in ways that are very different from one another from a physical point of view. As stated before, it corresponds to the modeling of an MDOF potential system whose damping matrix $D \neq 0$ is arbitrary—the matrix $D = S + G$ need not be symmetric and need not commute with K, as required in Ref. [2]. But it can model just as well an MDOF potential system subjected to a gyroscopic force described by the matrix G and a damping force represented by the symmetric matrix S. Examples of such systems are a damped whirring rod, and the damped motion of low-orbit satellites in a rotating frame of reference. More precisely, given a linear damped MDOF potential system, A_1, with an arbitrary non-zero damping matrix D, there is a corresponding linear damped MDOF gyroscopic potential system, A_2, with symmetric damping matrix S and gyroscopic matrix G such that both these systems have the same response to any given set of initial conditions and/or any given set of external forces. In other words, these two dynamical systems are described by the same governing equation of motion given in Eq. (7). When D tends to S, then G tends to 0, and the damped gyroscopic system becomes degenerate, since the gyroscopic force then tends to zero.

This paper therefore presents a unified view of these two physically dissimilar systems—damped potential systems and damped gyroscopic systems—which have been usually labeled in the current mechanics literature to date as belonging to different categories of dynamical systems due to the widely different nature of the physical forces that act in each of them (see, e.g., Ref. [6]). Since they share the same equation of motion the latter system can be thought of as the dual of the former, and vice versa. Our exploration into the dynamical uncoupling of these systems into independent, smaller-dimensional subsystems (along with the necessary and sufficient conditions required for such an uncoupling to occur) is therefore applicable to each member of this pair of dual systems, though throughout this paper we will use the notion of a damped potential system as a vehicle to introduce our results. It is hoped that the reader will keep in mind the closely allied dual notion of a damped gyroscopic potential system (with a symmetrical damping matrix S), which exhibits an identical dynamical behavior to that of the damped potential system that is continually referred to in the paper.

Our overall goal is to find a real orthogonal matrix Q (and the conditions under which it might exist) so that a real change of coordinates $x = Qp$ transforms Eq. (7) into a canonical (simplest) form that is maximally uncoupled. We shall call p the principal coordinate. Note that p is obtained from x by a linear transformation through the matrix Q, which physically represents simply a rotation or a reflection.

Let us assume, for a moment, that such a real orthogonal matrix Q exists. Upon multiplication of Eq. (7) from the left by Q^T, the use of this coordinate change yields the real system of equations given by $(G \neq 0)$

$$\ddot{p} + Q^T SQ\,\dot{p} + Q^T GQ\,\dot{p} + Q^T KQp = Q^T f(t) \tag{8}$$

The simplest (canonical) form that this equation could take would occur when the matrices $Q^T SQ$, $Q^T GQ$, and $Q^T KQ$ are diagonal. If this were possible, Eq. (8) would decompose into an uncoupled system of n real independent (second-order differential) equations. However, no real linear transformation, leaving aside an orthogonal one, can accomplish this since the eigenvalues of the skew-symmetric matrix G are purely imaginary; they come in complex conjugate pairs. Thus, uncoupling this system of equations to obtain n uncoupled equations is impossible, and such a system cannot have classical normal modes. In other words, the system cannot, in general, be uncoupled into n independent subsystems, each of which is a single-degree-of-freedom subsystem.

In what follows, Eq. (8) will be important for us, because we will show that an orthogonal matrix Q *can* be found such that the dynamical system given in Eq. (8) can be uncoupled into independent subsystems, each of which has no more than two degrees-of-freedom when the matrices S, G, and K satisfy certain necessary and sufficient conditions. These conditions, under which this uncoupling is guaranteed, are explicitly obtained. But before we do this, we present and prove two central theorems, which unify the behavior of the two physically disparate categories of dynamical systems mentioned earlier, and upon which our results will rest.

The structure of this paper is as follows. In Sec. 2, we present two theorems in linear algebra, that obtain the necessary and sufficient conditions for a linear damped MDOF potential system (with a general damping matrix) to be uncoupled; a total of seven conditions are obtained. Section 3 develops analytical conditions for the uncoupling of such systems based on our results in Sec. 2. The number of necessary and sufficient conditions is shown to reduce to just three for systems that are commonly encountered in aerospace, civil, and mechanical engineering as well as in nature. Section 4 deals with positing a useful general form for the damping matrix that further reduces the number of conditions for uncoupling of such MDOF systems to two. Several numerical examples are provided in Secs. 3 and 4 to illustrate the analytical results. Section 5 presents the main conclusions of this study.

2 Central Theorems

In this section, we obtain two results that provide the necessary and sufficient conditions for the simultaneous quasi-diagonalization of the matrices $Q^T SQ$, $Q^T GQ$, $Q^T KQ$ in Eq. (8). This leads to a maximal uncoupling of the MDOF system into independent subsystems, each with either a single-degree-of-freedom or with two degrees-of-freedom.

Theorem 1. *Let $K = K^T$, $S = S^T$, and $G = -G^T$ be $n \times n$ real matrices, and let $\text{Rank}(G) = 2m \leq n$. The necessary and sufficient conditions for a real orthogonal matrix Q to exist such that*

$$Q^T KQ = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \tag{9}$$

and

$$Q^T SQ = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \tag{10}$$

and

$$Q^T GQ = \Gamma = \text{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right) \tag{11}$$

are that the symmetric matrices

$$K,\ S,\ G^2,\ GKG,\ GSG \tag{12}$$

commute pairwise. The eigenvalues of K and S are $\lambda_i (i = 1, 2, \ldots, n)$ and $\sigma_i (i = 1, 2, \ldots, n)$, respectively.

Proof. That the last three matrices in Eq. (12) are symmetric, are simple to check.

(1) *Necessity*. Assume first that a real orthogonal matrix Q exists such that an orthogonal reduction to the forms given in Eqs. (9)–(11) occurs. We need to show that this implies that the symmetric matrices given in Eq. (12) commute pairwise. But, this follows quite directly, because the matrices in Eq. (12) are each orthogonally similar to diagonal matrices, and therefore must commute pairwise. For example, assuming that Q exists such that Eqs. (9)–(11) hold, we find that

$$Q^T(GKG)Q = (Q^TGQ)(Q^TKQ)(Q^TGQ) = \Gamma\Lambda\Gamma$$
$$= -\text{diag}(\beta_1^2\lambda_2, \beta_1^2\lambda_1, \ldots, \beta_m^2\lambda_{2m}, \beta_m^2\lambda_{2m-1}, 0, \ldots, 0)$$

so that $GKG = Q(\Gamma\Lambda\Gamma)Q^T$, and the matrix $(\Gamma\Lambda\Gamma)$ is diagonal. Similarly,

$$Q^T(GSG)Q = \Gamma\Sigma\Gamma = -\text{diag}(\beta_1^2\sigma_2, \beta_1^2\sigma_1, \ldots, \beta_m^2\sigma_{2m}, \beta_m^2\sigma_{2m-1}, 0, \ldots, 0)$$

so that $GSG = Q(\Gamma\Sigma\Gamma)Q^T$, and the matrix $(\Gamma\Sigma\Gamma)$ is again diagonal. Thus, the commutator

$$[GKG, GSG] := (GKG)(GSG) - (GSG)(GKG)$$
$$= Q\{(\Gamma\Lambda\Gamma)(\Gamma\Sigma\Gamma) - (\Gamma\Sigma\Gamma)(\Gamma\Lambda\Gamma)\}\, Q^T = 0$$

since the diagonal matrices $(\Gamma\Lambda\Gamma)$ and $(\Gamma\Sigma\Gamma)$ inside the curly brackets commute.

(2) *Sufficiency*. We next assume that the matrices K, S, G^2, GKG, GSG commute pairwise and show that an orthogonal matrix Q exists such that Eqs. (9)–(11) are true.

Let $\Lambda(G) = (\pm i\beta_1, \ldots, \pm i\beta_m, 0, \ldots, 0), \beta_j \neq 0, j = 1, 2, \ldots, m,$ be the spectrum (denoted by $\Lambda[\cdot]$) of the skew-symmetric matrix G. Then $\Lambda(G^2) = (-\beta_1^2, -\beta_1^2, \ldots, -\beta_m^2, -\beta_m^2, 0, \ldots, 0)$. Since the symmetric matrices $K, S, G^2, GKG,$ and GSG commute pairwise, according to a well-known result (see, e.g., Ref. [5]), they have n common linearly independent eigenvectors.

With no loss of generality, let q_1 be a (real) unit eigenvector such that

$$G^2q_1 = -\beta_1^2 q_1, \beta_1 \neq 0$$
$$Kq_1 = \lambda_1 q_1, Sq_1 = \sigma_1 q_1$$
$$GKGq_1 = \mu_1 q_1$$

and

$$GSGq_1 = \eta_1 q_1$$

where $\lambda_1, \sigma_1, \mu_1,$ and η_1 are real numbers, which could be zero. Premultiplying each of the last two equations by G gives $G^2KGq_1 = \mu_1 Gq_1$ and $G^2SGq_1 = \eta_1 Gq_1$. Since $G^2K = KG^2$ (because G^2 and K commute) and $G^2S = SG^2$ (because G^2 and S commute), we then get $KGG^2q_1 = \mu_1 Gq_1$ and $SGG^2q_1 = \eta_1 Gq_1$. Furthermore, because $G^2q_1 = -\beta_1^2 q_1$, these two relations become

$$K(Gq_1) = -\mu_1\beta_1^{-2}(Gq_1)$$

and

$$S(Gq_1) = -\eta_1\beta_1^{-2}(Gq_1)$$

From this, it follows that $-Gq_1$ is an eigenvector of both K and S. Also, since $\|Gq_1\| = \sqrt{q_1^TG^TGq_1} = \sqrt{-q_1^TG^2q_1} = \sqrt{\beta_1^2 q_1^Tq_1} = \beta_1 \neq 0$, we see that the vector $q_2 := -\beta_1^{-1}Gq_1$ is a unit eigenvector of the matrices K and S, corresponding to the eigenvalue $\lambda_2 := -\mu_1\beta_1^{-2}$ and $\sigma_2 := -\eta_1\beta_1^{-2}$, respectively. We therefore obtain $Kq_j = \lambda_j q_j$, $j = 1, 2$, and $Sq_j = \sigma_j q_j, j = 1, 2$. Furthermore, because G is skew-symmetric, $q_1^Tq_2 = -\beta_1^{-1}q_1^TGq_1 = 0$, i.e., the unit vectors q_1 and q_2 are orthogonal.

Now using q_1 and q_2 as the first and second columns, we form an orthogonal matrix $Q_1 = \begin{bmatrix} q_1 & q_2 & q_3 & \ldots & q_n \end{bmatrix}$, whose remaining columns can be chosen arbitrarily provided $Q_1^TQ_1 = I_n$. We next determine the structure of the symmetric matrices $Q_1^TKQ_1 := [q_j^TKq_k]$ and $Q_1^TSQ_1 := [q_j^TSq_k]$, and the structure of the skew-symmetric matrix $Q_1^TGQ_1 := [q_j^TGq_k]$.

We see that for $k = 1, 2, \ldots, n,$ noting the orthogonality of the columns of Q_1, the elements of the first and second rows (columns) of $Q_1^TKQ_1$ are given by

$$q_1^TKq_k = q_k^TKq_1 = \lambda_1 q_k^Tq_1 = \lambda_1\delta_{1k} \text{ and } q_2^TKq_k = q_k^TKq_2 = \lambda_2 q_k^Tq_2 = \lambda_2\delta_{2k} \tag{13}$$

where δ_{jk} denotes the Kronecker delta. Similarly, for $k = 1, 2, \ldots, n,$ the first two rows (columns) of $Q_1^TSQ_1$ are given by

$$q_1^TSq_k = q_k^TSq_1 = \sigma_1 q_k^Tq_1 = \sigma_1\delta_{1k} \text{ and } q_2^TSq_k = q_k^TSq_2 = \sigma_2 q_k^Tq_2 = \sigma_2\delta_{2k} \tag{14}$$

For $k = 1, 2, \ldots, n,$ noting that $Gq_1 = -\beta_1 q_2$ and $Gq_2 = -\beta_1^{-1}G^2q_1 = \beta_1 q_1$, the elements of the first and second rows (columns) of $Q_1^TGQ_1$ are given, respectively, by the relations

$$q_1^TGq_k = -q_k^TGq_1 = \beta_1 q_k^Tq_2 = \beta_1\delta_{2k} \text{ and } q_2^TGq_k = -q_k^TGq_2 = -\beta_1 q_k^Tq_1 = -\beta_1\delta_{1k} \tag{15}$$

From Eqs. (13)–(15), the structure of each of the three matrices $Q_1^TKQ_1$, $Q_1^TSQ_1$, and $Q_1^TGQ_1$ is hence found to be as follows:

$$Q_1^TKQ_1 = \left[\begin{array}{cc|c} \lambda_1 & 0 & \\ 0 & \lambda_2 & 0 \\ \hline \multicolumn{2}{c|}{0} & K_{n-2} \end{array}\right]$$

$$Q_1^TSQ_1 = \left[\begin{array}{cc|c} \sigma_1 & 0 & \\ 0 & \sigma_2 & 0 \\ \hline \multicolumn{2}{c|}{0} & S_{n-2} \end{array}\right]$$

and

$$Q_1^TGQ_1 = \left[\begin{array}{cc|c} 0 & \beta_1 & \\ -\beta_1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & G_{n-2} \end{array}\right]$$

Since the $(n-2)$-dimensional matrices K_{n-2}, S_{n-2}, and G_{n-2} satisfy the same conditions as K, S, and G, this procedure continues in the same manner and after m steps we conclude that there exists an orthogonal matrix $\hat{Q}$ such that

$$\hat{Q}^TG\hat{Q} = \text{diag}\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right)$$

$$\hat{Q}^TK\hat{Q} = \text{diag}(\lambda_1, \ldots, \lambda_{2m}, K_{n-2m})$$

and

$$\hat{Q}^TS\hat{Q} = \text{diag}(\sigma_1, \ldots, \sigma_{2m}, S_{n-2m})$$

where K_{n-2m} and S_{n-2m} are $(n-2m)$-dimensional symmetric matrices that commute, i.e., $K_{n-2m}S_{n-2m} = S_{n-2m}K_{n-2m}$. Since they commute, there exists an orthogonal matrix $\bar{Q}_{n-2m}$ of order $(n-2m)$ which reduces the matrices K_{n-2m} and D_{n-2m} simultaneously to diagonal forms. Consequently, the orthogonal matrix

$$Q = \hat{Q}\begin{bmatrix} I_{2m} & 0 \\ 0 & \bar{Q}_{n-2m} \end{bmatrix}$$

simultaneously reduces K, S, and G to the forms given in Eqs. (9)–(11). ∎

THEOREM 2. *Let $K = K^T$, $S = S^T$, and $G = -G^T$ be $n \times n$ real matrices, and let* Rank$(G) = 2m \leq n$. *The necessary and sufficient conditions for a real orthogonal matrix* Q *to exist such that*

$$Q^TKQ = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \tag{16}$$

and

$$Q^TSQ = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \tag{17}$$

and

$$Q^TGQ = \Gamma = \mathrm{diag}\left(\beta_1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \ldots, \beta_m\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, 0, \ldots, 0\right) \quad (18)$$

are that the following seven commutation conditions be met:

$$[K, S] = 0 \quad (19)$$

$$[K, G^2] = 0, [K, GKG] = 0 \quad (20)$$

$$[S, G^2] = 0, [S, GSG] = 0 \quad (21)$$

and

$$[K, GSG] = 0, [S, GKG] = 0 \quad (22)$$

where the commutator of any two square matrices A and B is defined as $[A, B] := AB - BA$.

Proof. According to Theorem 1, the necessary and sufficient conditions for the existence of a real orthogonal matrix Q to simultaneously reduce S, G, and K to quasi-diagonal form is that the five matrices K, S, G^2, GKG, GSG in Eq. (12) commute pairwise. There are therefore a total of $C_2^5 = 10$ necessary and sufficient conditions in all. Each of these pairwise commutation conditions can be expressed in the commutator notation. However, the ten pairwise commutation relations that they generate are not all independent of one another. Three of them follow from the remaining seven listed in Eqs. (19)–(22), as we now show.

(1) The condition that the matrices G^2 and GKG commute, which can be written as $[G^2, GKG] = 0$, follows from the condition that G^2 also commutes with K ($[K, G^2] = 0$). Using the latter condition, we get $G^2(GKG) = GG^2KG = GKG^2G = (GKG)G^2$.
(2) Replacing K by S in (1) above, we get $[G^2, GSG] = 0$, upon using the condition that G^2 and S commute ($[S, G^2] = 0$).
(3) Finally, the condition $[GKG, GSG] = 0$, follows upon using the two conditions $[G^2, K] = 0$, $[G^2, S] = 0$ and $[K, S] = 0$, since $(GKG)\ (GSG) = GKG^2SG = GG^2KSG = GG^2SKG = GSG^2KG = (GSG)\ (GKG)$.

This leaves us with the seven pairwise commutation conditions, which are listed in Eqs. (19)–(22). ∎

It is this *Theorem* that we shall invoke in what follows. When the forms in Eqs. (16)–(18) are obtained we shall refer to this as the *simultaneous quasi-diagonalization of the matrices S, G, and K* by the (orthogonal) matrix Q. We next give two lemmas that will be used later.

Lemma 2. *Let A_1 and A_2 be two n by n matrices. If there exists an orthogonal matrix Q such that $A_1 = Q\Lambda_1Q^T$ and $A_2 = Q\Lambda_2Q^T$ where Λ_1 and Λ_2 are diagonal matrices, then* $[A_1, A_2] = 0$.

Proof. The product $A_1A_2 = Q\Lambda_1Q^TQ\Lambda_2Q^T = Q\Lambda_1\Lambda_2Q^T := Q\Lambda_{12}Q^T$, and similarly, $A_2A_1 = Q\Lambda_2\Lambda_1Q^T := Q\Lambda_{21}Q^T$, where the diagonal matrix $\Lambda_{12} = \Lambda_{21}$ since diagonal matrices commute with each other. Hence, the commutator $[A_1, A_2] = A_1A_2 - A_2A_1 = Q\Lambda_{12}Q^T - Q\Lambda_{21}Q^T = Q[\Lambda_{12} - \Lambda_{21}]Q^T = 0$. ∎

Lemma 3. *Let S and K be any two 2 by 2 symmetric matrices that commute with one another, i.e., $[S, K] = 0$. Then the matrices S and K satisfy the seven conditions given by Eqs. (19)–(22) for any arbitrary 2 by 2 skew-symmetric matrix G.*

Proof. Since S and K are symmetric and they commute, there exists a 2 by 2 orthogonal matrix Q such that $S = Q\Sigma Q^T$ and $K = Q\Lambda Q^T$ where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2)$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ [5]. Furthermore, every(any) 2 by 2 skew-symmetric matrix G can be written (with no loss of generality) as

$$G = \beta Q\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}Q^T := \beta QJ_2Q^T$$

where J_2 denotes the skew-symmetric matrix on the right-hand side of the first equality, and β is a suitable real number.

Hence,

$S = Q\Sigma Q^T$, $K = Q\Lambda Q^T$

$G^2 = \beta^2QJ_2J_2Q^T = Q(-\beta^2I_2)Q^T$, where I_2 is the 2 by 2 identity matrix,

$$GSG = \beta^2QJ_2\Sigma J_2Q^T = Q\begin{bmatrix}-\beta^2\sigma_2 & 0\\ 0 & -\beta^2\sigma_1\end{bmatrix}Q^T$$

and

$$GKG = \beta^2QJ_2\Lambda J_2Q^T = Q\begin{bmatrix}-\beta^2\lambda_2 & 0\\ 0 & -\beta^2\lambda_1\end{bmatrix}Q^T$$

Using *Lemma 2* with $n = 2$, each of the commutation requirements in Eqs. (19)–(22) are satisfied. Hence, the result follows. ∎

Remark 1. Lemma 3 shows, in particular, that if S and K are any 2 by 2 symmetric matrices that commute ($[K, S] = 0$), and if G is any arbitrary 2 by 2 skew-symmetric matrix, then $[K, GKG] = [S, GSG] = [K, GSG] = [S, GKG] = 0$. We will use this observation later on. ∎

Theorem 2 allows us to trivially obtain a previous result, which we state here as a corollary since we will be using it later on.

Corollary 1. *Let $K = K^T$ and $G = -G^T \neq 0$ be n by n real matrices, and let* $\mathrm{Rank}(G) = 2m \leq n$. *The necessary and sufficient conditions that there exists a real orthogonal matrix Q such that*

$$Q^TKQ = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \quad (23)$$

$$Q^TGQ = \Gamma = \mathrm{diag}\left(\beta_1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \ldots, \beta_m\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, 0, \ldots, 0\right) := \mathrm{diag}(\beta_1J_2, \ldots, \beta_mJ_2, 0, \ldots 0) \quad (24)$$

$$Q^TG^2Q = \Gamma^2 = -\mathrm{diag}(\beta_1^2, \beta_1^2, \ldots, \beta_m^2, \beta_m^2, 0, \ldots, 0) \quad (25)$$

and

$$Q^T(GKG)Q = (\Gamma\Lambda\Gamma) = -\mathrm{diag}(\beta_1^2\lambda_2, \beta_1^2\lambda_1, \ldots, \beta_m^2\lambda_{2m}, \beta_m^2\lambda_{2m-1}, 0, \ldots, 0) \quad (26)$$

where all the λ_j's and β_j's are real numbers, are

$$[K, G^2] = 0, \text{ or } KG^2 = G^2K \quad (27)$$

and

$$[K, GKG] = 0, \text{ or } (KG)^2 = (GK)^2 \quad (28)$$

Proof. Application of *Theorem 2* with $S = 0$ gives the result. (also see Refs. [7,8] for earlier alternative proofs.) As before, J_2 denotes the 2 by 2 skew-symmetric matrix shown on the right-hand side of Eq. (24). Equations (25) and (26) follow from Eqs. (23) and (24). ∎

3 Uncoupling of Damped Potential MDOF Systems

We now consider uncoupling of the MDOF system described by Eq. (7). Recall that the results are equally applicable to a damped gyroscopic MDOF system (with a symmetric damping matrix) that "shadows" the damped potential system and is its dual.

Result 1. Consider the system described by Eq. (7) in which $\mathrm{Rank}(G) = 2m \le n$. Then the conditions

$$[K, S] = 0 \tag{29}$$

$$[K, G^2] = 0, \quad [K, GKG] = 0 \tag{30}$$

$$[S, G^2] = 0, \quad [S, GSG] = 0 \tag{31}$$

and

$$[K, GSG] = 0, \quad [S, GKG] = 0 \tag{32}$$

are *necessary* and *sufficient conditions* for Eq. (7) to be decomposed by an orthogonal congruence transformation $x = Qp$ into uncoupled, independent subsystems, m of which are quasi-diagonalized two degrees-of-freedom and $n - 2m$ of which are single-degree-of-freedom subsystems. The uncoupled equations in the principal coordinates p have the form

$$\ddot{p} + \Sigma\dot{p} + \Gamma\dot{p} + \Lambda p = Q^T f(t) \tag{33}$$

where

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{34}$$

$$\Gamma = \mathrm{diag}\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right)$$
$$:= \mathrm{diag}(\beta_1 J_2, \ldots, \beta_m J_2, 0, \ldots, 0) \tag{35}$$

and

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{36}$$

Proof. As shown before, by using a real orthogonal transformation $x = Qp$ Eq. (7) becomes Eq. (8), which is (we repeat it for convenience)

$$\ddot{p} + Q^T SQ\dot{p} + Q^T GQ\dot{p} + Q^T KQp = Q^T f(t)$$

According to *Theorem 2*, the conditions in Eqs. (29)–(32) are both *necessary* and *sufficient* for the existence of a real orthogonal matrix Q such that $Q^T SQ = \Sigma$, $Q^T GQ = \Gamma$, and $Q^T KQ = \Lambda$, from which we get Eq. (33) with Σ, Γ, and Λ, as in Eqs. (34)–(36). ∎

Remark 2. When $G = 0$, the conditions given in Eqs. (30)–(32) trivially hold, leaving only the condition $[K, S] = 0$. Thus, the conditions in Result 1 reduce to the single, well-known, necessary, and sufficient condition for the complete decoupling (diagonalizing) of damped potential systems (Caughey–O'Kelly [2]) in which the damping matrix S is symmetric and commutes with the symmetric matrix K.

On the other hand, when $S = 0$, the conditions in *Result 1* reduce to the two conditions given in Eq. (30) which are, as shown recently in Ref. [8], necessary and sufficient for quasi-diagonalization of conservative gyroscopic systems (see *Corollary 1*). ∎

Corollary 2. *The system given in Eq. (7) with two degrees-of-freedom can be transformed by an orthogonal congruence transformation to the form Eq. (33) if and only if* $[K, S] = 0$.

Proof. See *Lemma 3*. ∎

Corollary 3. *Let,* $G = -G^T$, $K = K^T$, $S = aI + bK$, *and* $\mathrm{Rank}(G) = 2m \le n$. *Then the conditions* $[K, G^2] = 0$, *and* $[K, GKG] = 0$, *are necessary and sufficient for Eq. (7) to be transformed by an orthogonal congruence transformation* $x = Qp$ *to the form*

$$\ddot{p} + (aI + b\Lambda)\dot{p} + \Gamma\dot{p} + \Lambda p = Q^T f(t)$$

with Γ *and* Λ *as in Eqs. (35) and (36). Note that the structure of the matrix S here corresponds to the notion of Rayleigh damping* [9].

Proof. The other necessary and sufficient conditions given in *Result 1* for an orthogonal matrix Q to exist so that S, G, and K are simultaneously quasi-diagonalized are automatically satisfied when the two conditions in the statement of this corollary are satisfied. For example,

$$\begin{aligned}[S, GSG] &= [aI + bK, GSG] = a[I, GSG] + b[K, GSG] \\ &= b[K, G(aI + bK)G] \\ &= ab[K, G^2] + b^2[K, GKG] = 0\end{aligned}$$ ∎

Corollary 4. *Let* $S = S^T$, $G = -G^T \ne 0$, $K = K^T$, *and* $\mathrm{Rank}(G) = 2m \le n$. *If the matrices S, G,* and *K, commute pairwise, i.e.,*

$$[K, S] = 0, [S, G] = 0, [K, G] = 0 \tag{37}$$

then there exists a real linear orthogonal change of coordinates that transforms the dynamical system given in Eq. (7) to the form given in Eq. (33) with

$$\Sigma = \mathrm{diag}(\sigma_1 I_2, \ldots, \sigma_m I_2, \sigma_{2m+1}, \ldots, \sigma_n) \tag{38}$$

$$\Gamma = \mathrm{diag}\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \ldots \quad \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad 0, \quad \ldots \quad 0\right) \tag{39}$$

and

$$\Lambda = \mathrm{diag}(\lambda_1 I_2, \ldots, \lambda_m I_2, \lambda_{2m+1}, \ldots, \lambda_n) \tag{40}$$

Proof. If the matrices S, G, and K commute pairwise then the conditions in Eqs. (29)–(32) are satisfied, and, according to *Result 1*, there exists a real orthogonal transformation Q which transforms Eq. (7) to the form given in Eq. (33). Moreover, the last two conditions in Eq. (37) correspond to the conditions $[\Sigma, \Gamma] = 0$ and $[\Lambda, \Gamma] = 0$ (see *Lemma 2*) which require $\sigma_1 = \sigma_2$, $\sigma_3 = \sigma_4$, $\ldots, \sigma_{2m-1} = \sigma_{2m}$ and $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4, \ldots, \lambda_{2m-1} = \lambda_{2m}$, because $\beta_j \ne 0$. We then get Eqs. (38) and (40). ∎

The following results are also consequences of *Result 1*. They refer to commonly occurring situations in engineered structural and mechanical systems as well as in naturally occurring systems, and they contain fewer commutativity conditions.

Result 2. Let, $S = S^T$, $G = -G^T$, $K = K^T$, and $\mathrm{Rank}(G) = 2m \le n$. If all non-zero eigenvalues of the skew-symmetric matrix G are distinct, then there exists a real linear orthogonal change of coordinates that transforms the system given in Eq. (7) to the form given in Eqs. (33)–(36) if and only if the matrices S, G^2, and K commute pairwise, i.e.,

$$[S, G^2] = 0, [K, G^2] = 0, \text{ and } [K, S] = 0 \tag{41}$$

Proof. Let Q be a real orthogonal matrix such that [5]

$$G = Q\begin{bmatrix} \hat{G} & 0 \\ 0 & 0_{n-2m} \end{bmatrix}Q^T \tag{42}$$

and

$$K = Q\begin{bmatrix} \hat{K} & \bar{K} \\ \bar{K}^T & \hat{\hat{K}} \end{bmatrix}Q^T \tag{43}$$

and

$$S = Q\begin{bmatrix} \hat{S} & \bar{S} \\ \bar{S}^T & \hat{\hat{S}} \end{bmatrix} Q^T \tag{44}$$

where

$$\hat{G} = \text{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) := \text{diag}(\hat{G}_{jj})_{j=1}^m \tag{45}$$

and 0_{n-2m} is $(n-2m)$-dimensional zero matrix, $\hat{K}(\hat{S})$ and $\hat{\hat{K}}(\hat{\hat{S}})$ are $2m$ and $(n-2m)$ dimensional symmetric matrices respectively, $\bar{K}(\bar{S})$ is $2m$ by $(n-2m)$ matrix, and $\hat{G}_{jj} = \beta_j J_2$. From Eqs. (42) and (45), we see that $G^2 = -Q\,\text{diag}(\beta_1^2 I_2, \ldots, \beta_m^2 I_2, 0, \ldots, 0)Q^T$ where I_2 denotes the 2 by 2 identity matrix. Then, the conditions $[S, G^2] = 0$ and $[K, G^2] = 0$ yield $\bar{S} = 0$ and $\bar{K} = 0$, because $\hat{G}^2$ is nonsingular, and also

$$\hat{S}\hat{G}^2 = \hat{G}^2\hat{S},\ \hat{K}\hat{G}^2 = \hat{G}^2\hat{K} \tag{46}$$

Noting that $\hat{G}^2 = -\text{diag}(\beta_1^2 I_2, \ldots, \beta_m^2 I_2)$, and partitioning the symmetric matrices $\hat{K}$ and $\hat{S}$ as $\hat{K} = [\hat{K}_{jk}]_{j,k=1}^m$ and $\hat{S} = [\hat{S}_{jk}]_{j,k=1}^m$ with two-dimensional sub-matrices $\hat{K}_{jk}$ and $\hat{S}_{jk}$ conditions (46) become

$$[\beta_k^2 \hat{S}_{jk}]_{j,k=1}^m = [\beta_j^2 \hat{S}_{jk}]_{j,k=1}^m,\ [\beta_k^2 \hat{K}_{jk}]_{j,k=1}^m = [\beta_j^2 \hat{K}_{jk}]_{j,k=1}^m$$

or $(\beta_j^2 - \beta_k^2)\hat{S}_{jk} = 0$ and $(\beta_j^2 - \beta_k^2)\hat{K}_{jk} = 0$ which yield $\hat{S}_{jk} = 0$ and $\hat{K}_{jk} = 0$ for $j \neq k$ since, by assumption, all the numbers $\beta_1, \ldots, \beta_m$, are distinct. Thus, the matrices K and S that satisfy the first two conditions in Eq. (41) must be of the forms

$$K = Q\begin{bmatrix} \text{diag}(\hat{K}_{jj})_{j=1}^m & 0 \\ 0 & \hat{\hat{K}} \end{bmatrix} Q^T \tag{47}$$

and

$$S = Q\begin{bmatrix} \text{diag}(\hat{S}_{jj})_{j=1}^m & 0 \\ 0 & \hat{\hat{S}} \end{bmatrix} Q^T \tag{48}$$

where $\hat{K}_{jj}(\hat{S}_{jj})$, $j = 1, \ldots, m$, are 2 by 2 symmetric matrices and $\hat{\hat{K}}(\hat{\hat{D}})$ is an $(n-2m)$-dimensional symmetric matrix.

The condition $[K, S] = 0$ requires $[\hat{K}_{jj}, \hat{S}_{jj}] = 0$, $j = 1, \ldots, m$, and $[\hat{\hat{K}}, \hat{\hat{S}}] = 0$. Observe that from Eqs. (42), (47), and (48)

$$GSG = Q\begin{bmatrix} \text{diag}(G_{jj}\hat{S}_{jj}G_{jj})_{j=1}^m & 0 \\ 0 & 0_{n-2m} \end{bmatrix} Q^T \tag{49}$$

and

$$GKG = Q\begin{bmatrix} \text{diag}(G_{jj}\hat{K}_{jj}G_{jj})_{j=1}^m & 0 \\ 0 & 0_{n-2m} \end{bmatrix} Q^T \tag{50}$$

Since for each of the 2 by 2 matrices K_{jj} and S_{jj}, we have $[\hat{K}_{jj}, \hat{S}_{jj}] = 0$, $j = 1, \ldots, m$, from *Remark 1* we know that for $j = 1, \ldots, m$

$$[K_{jj}, G_{jj}K_{jj}G_{jj}] = [S_{jj}, G_{jj}S_{jj}G_{jj}] = [K_{jj}, G_{jj}S_{jj}G_{jj}] = [S_{jj}, G_{jj}K_{jj}G_{jj}] = 0$$

Noting the block-diagonal structure of the matrices in Eq. (42) and in Eqs. (47)–(50), we therefore find that the four conditions

$$[K, GKG] = [S, GSG] = [K, GSG] = [S, GKG] = 0 \tag{51}$$

are satisfied.

Then, *Result 2* now follows from *Result 1*. ■

Remark 3. Result 1 gives seven n&s conditions for the existence of an orthogonal matrix Q that simultaneously quasi-diagonalizes the matrices S, G, and K. *Result 2* shows that an orthogonal matrix Q exists such that S, G, and K can be quasi-diagonalized when the non-zero eigenvalues of G are distinct with the number of necessary and sufficient conditions reduces to just three from among the seven n&s relations given in *Result 1*. Therefore, *Result 2* shows that the presence of distinct eigenvalues in G *along with* the three conditions given in Eq. (41) cause the remaining n&s conditions stated in *Result 1*, namely, $[K, GKG] = 0$, $[S, GSG] = 0$, $[S, GKG] = 0$, $[K, GDG] = 0$ to be automatically satisfied. ■

Example 1. Consider the dynamical system

$$\ddot{x} + \underbrace{(S + G)}_{D}\dot{x} + Kx = f(t)$$

with

$$K = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, G = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}, S = \begin{bmatrix} 8 & -4 & 1 \\ -4 & 11 & 4 \\ 1 & 4 & 8 \end{bmatrix}$$

Since G is 3 by 3, one of its eigenvalues must be zero, and the other two must be imaginary complex conjugates of each other. Thus, all the eigenvalues of G are distinct and its rank is 2.

We next calculate

$$G^2 = \begin{bmatrix} -5 & -2 & -1 \\ -2 & -2 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

and find that

$$KG^2 = -9\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = G^2K,\ SG^2 = -\begin{bmatrix} 33 & 6 & 21 \\ 6 & 6 & -6 \\ 21 & -6 & 33 \end{bmatrix}$$
$$= G^2S,\ KS = 3\begin{bmatrix} 7 & -5 & 2 \\ -5 & 10 & 5 \\ 2 & 5 & 7 \end{bmatrix} = SK$$

The conditions in Eq. (41) are therefore satisfied, and according to *Result 2*, there exists a principal coordinate p in which this dynamical system, decomposes into one two degrees-of-freedom subsystem and one single-degree-of-freedom subsystem. Indeed, the coordinate change $x = Qp$ with

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

uncouples the dynamical system to the form

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 9\dot{p}_1 \\ 3\dot{p}_2 \end{bmatrix} + \sqrt{6}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 3p_1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} (f_1(t) + f_3(t))/\sqrt{2} \\ (f_1(t) + f_2(t) - f_3(t))/\sqrt{3} \end{bmatrix}$$
$$\ddot{p}_3 + 15\dot{p}_3 + 3p_3 = (f_1(t) - 2f_2(t) - f_3(t))/\sqrt{6}$$ ■

Result 3. Let $S = S^T$, $G = -G^T$, $K = K^T$, and $\text{Rank}(G) = 2m \leq n$. If all eigenvalues of the potential matrix K are distinct, then there exists a real linear orthogonal change of coordinates $x = Qp$ that transforms the system shown in Eq. (7) to the form given in Eqs. (33)–(36) if and only if the following conditions hold:

$$[K, G^2] = 0,\ [K, GKG] = 0,\ [K, S] = 0 \tag{52}$$

Proof. The necessity is obvious (see *Proof of Theorem 1*). To prove sufficiency, we assume that the conditions in Eq. (52) are satisfied. Then, according to *Corollary 1*, there exists a real orthogonal matrix

Q such that

$$Q^T G Q = \Gamma$$
$$= \mathrm{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \ldots, \quad \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad 0, \quad \ldots, \quad 0\right)$$

and

$$Q^T K Q = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

with all λ_j distinct. Furthermore, the condition $[K, S] = 0$ becomes $[\Lambda, Q^T S Q] = 0$, and from this, it follows that $Q^T S Q$ must be a diagonal matrix, since all the diagonal elements of Λ are distinct [5]. Hence, Q simultaneously diagonalizes S. ■

Remark 4. Since K has distinct eigenvalues and it commutes with S, the matrix S can always be expressed as a polynomial in the matrix K [5]. Since we have shown that an orthogonal Q exists that simultaneously quasi-diagonalizes S, G, and K, when Eq. (52) is satisfied, the seven necessary and sufficient conditions given in *Result 1* must be satisfied. If the conditions in Eq. (52) are satisfied, then it can be shown that the remaining four conditions given in *Theorem 2* are indeed satisfied. If, in addition, the matrix G has distinct non-zero eigenvalues, then the first condition in Eq. (52) implies the second [8], and the number of necessary and sufficient conditions reduces to two, namely, $[K, G^2] = 0$ and $[K, S] = 0$. ■

Result 3 has special relevance to physical systems because they often (typically) have stiffness (potential) matrices whose eigenvalues are distinct. In fact, symmetric matrices with distinct eigenvalues are dense in the set of stiffness matrices [5]. This means that for almost all stiffness matrices (except perhaps for those whose structure may be restricted on physical grounds, e.g., by reasons of symmetry) infinitesimal changes in the values of the elements of a stiffness matrix that has multiple eigenvalues will render its eigenvalues distinct [5]. The reason why stiffness matrices in most aerospace, civil, and mechanical systems—those that are engineered as well as those found in nature—can be taken to have distinct eigenvalues is that when modeling a physical system the elements of its stiffness matrix are, at best, good approximations of those of the actual physical system. These elements are generally found analytically (based on some assumptions on material properties, geometry, etc.) and/or experimentally. Hence, if the stiffness matrix in the modeling of a physical system has multiple eigenvalues, an infinitesimal change in the values of its (matrix) elements will make its eigenvalues distinct. And, in general, such infinitesimal changes will provide as good an approximation as before to the actual stiffness matrix of the physical system.

Remark 5. Consider Eq. (2) (which we repeat here, for convenience)

$$\ddot{x} + D\dot{x} + Kx = f(t) \tag{53}$$

When D is non-symmetric and non-defective, Ref. [4] provides a necessary and sufficient condition for uncoupling the system using a complex *equivalence* transformation. The condition is that $[K, D] = 0$.

If K has eigenvalues that are all distinct (which is the typical case) and the matrix D is non-symmetric (the symmetric case is handled in Ref. [2]), then there is *no* complex equivalence transformation that will simultaneously diagonalize these two matrices and uncouple the system. This is because if K has distinct eigenvalues and it commutes with D, then D can always be expressed as a polynomial in K of degree at most $(n-1)$ [5], and since K is symmetric, D must be too. ■

Having seen $[K, D] \neq 0$ when all the eigenvalues of K are distinct, we next consider the case when K has l distinct eigenvalues with multiplicities greater than 1, and the remaining eigenvalues are distinct with multiplicity one.

Remark 6. Suppose that in Eq. (53) the matrix K has l multiple (distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_l$, with multiplicities $n_1, n_2, \ldots, n_l$, respectively, and all of the remaining eigenvalues are distinct. Assume further that $[K, D] = 0$.

Since K is symmetric, there exists a real orthogonal matrix Q with

$$Q^T K Q = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_l I_{n_l}, \bar{\Lambda}_{n-r})$$

where $r = \sum_{i=1}^{l} n_i$ and $\bar{\Lambda}_{n-r}$ is the $(n-r)$-dimensional diagonal matrix with distinct eigenvalues. Since K and D commute, we have

$$Q^T D Q = \mathrm{diag}(\hat{D}_{n_1}, \hat{D}_{n_2}, \ldots, \hat{D}_{n_l}, \bar{\hat{D}}_{n-r})$$

where $\hat{D}_{n_i}$, $i = 1, 2, \ldots, l$ are n_i by n_i, non-symmetric matrices, and $\bar{\hat{D}}_{n-r}$ is a real $(n-r)$-dimensional diagonal matrix. *If* D is non-defective, then Ref. [4] guarantees (see *Remark 5*) that a complex equivalence transformation exists that will simultaneously diagonalize K and D; a non-symmetric matrix D may or may not be defective.

On the other hand, writing $\hat{D}_{n_i} = \hat{S}_{n_i} + \hat{G}_{n_i}$, $i = 1, 2, \ldots, l$, where $\hat{S}_{n_i}$ and $\hat{G}_{n_i}$ are the symmetric and skew-symmetric parts of $\hat{D}_{n_i}$ and applying *Result 1*, we see that there exists a real orthogonal matrix that simultaneously quasi-diagonalizes both K and D if and only if

$$\hat{S}_{n_i}\hat{G}_{n_i}^2 = \hat{G}_{n_i}^2\hat{S}_{n_i} \text{ and } (\hat{S}_{n_i}\hat{G}_{n_i})^2 = (\hat{G}_{n_i}\hat{S}_{n_i})^2, \; i = 1, 2, \ldots, l \tag{54}$$

We do not require D to be non-defective.

When Eq. (54) is satisfied, upon using the coordinate transformation $x = Qp$ the MDOF system uncouples to yield independent subsystems of which $(n-2m)$ are single-degree-of-freedom and m are two degrees-of-freedom; here $2m$ is the rank of the skew-symmetric part of D.

We note that $[\hat{S}_{n_i}, \lambda_i I_{n_i}] = \lambda_i[\hat{S}_{n_i}, {}_i I_{n_i}] = 0$, $i = 1, 2, \ldots, l$. Therefore, when $n_i = 2$, $i = 1, 2, \ldots, l$, by *Lemma 3* the above conditions in Eq. (54) are automatically satisfied.

Thus, when $[K, D] = 0$ and the multiplicities of the eigenvalues of K are at most two we are guaranteed to have an orthogonal transformation Q such that the coordinate transformation $x = Qp$ uncouples the dynamical system in Eq. (53) into subsystems that have at most two degrees-of-freedom.

In practical engineering applications, one finds that in MDOF structural and mechanical systems the number of distinct repeated eigenvalues l is generally less than 3 or 4, and often just 1. ■

Example 2. We illustrate the result obtained in *Remark 6* by a simple example in which $l = 1$ and $n_1 = 2$, i.e., there is only one eigenvalue with multiplicity two, and all the other eigenvalues are distinct.

Consider the four degrees-of-freedom system described by Eq. (53) in which (for brevity, the numerical values are shown only up to four decimal figures)

$$K = \begin{bmatrix} 1.7715 & -0.8960 & -0.6389 & -1.0287 \\ -0.8960 & 3.1842 & 1.8855 & 1.1947 \\ -0.6389 & 1.8855 & 2.6726 & 0.8518 \\ -1.0287 & 1.1947 & 0.8518 & 2.3716 \end{bmatrix} \tag{55}$$

and

$$D = \begin{bmatrix} 0.0249 & 0.0069 & -0.0146 & -0.0035 \\ -0.0211 & 0.0063 & 0.0290 & -0.0016 \\ 0.0134 & 0.0290 & 0.0038 & 0.0118 \\ 0.0036 & 0.0194 & -0.0092 & 0.0250 \end{bmatrix} \tag{56}$$

A small computation shows that $[K, D] = 0$. Following the steps in *Remark 6*, we find that the matrix:

$$Q = \begin{bmatrix} -0.5000 & -0.6301 & -0.4983 & -0.3235 \\ 0.5000 & -0.4901 & -0.3020 & 0.6470 \\ -0.5000 & 0.4901 & -0.4681 & 0.5392 \\ 0.5000 & -0.3501 & 0.6644 & 0.4313 \end{bmatrix}$$

diagonalizes K so that

$$Q^T KQ = \text{diag}(1, 1, 2, 6) = \text{diag}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}\right):$$
$$= \text{diag}(I_2, \bar{\Lambda}_2)$$

showing that $\lambda_1 = 1$ is a multiple eigenvalue of K with multiplicity two, and $\lambda_2 = 2$ and $\lambda_3 = 6$ are the other two (distinct) eigenvalues. As mentioned earlier, this corresponds to $l = 1$ in *Remark 6*, with $n_1 = 2$. We also have

$$Q^T DQ = \text{diag}\left(\begin{bmatrix} 0 & 0.05 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.02 & 0 \\ 0 & 0.04 \end{bmatrix}\right): = \text{diag}(\hat{D}_1, \bar{\hat{D}}_2)$$

showing that 0 is a multiple eigenvalue of the matrix D with multiplicity two, the other two eigenvalues being 0.02 and 0.04.

Using the coordinate transformation $x = Qp$, Eq. (53) then becomes

$$\ddot{p} + \underbrace{\begin{bmatrix} \hat{D}_1 & 0 \\ 0 & \bar{\hat{D}}_2 \end{bmatrix}}_{\hat{D}} \dot{p} + \underbrace{\begin{bmatrix} I_2 & 0 \\ 0 & \bar{\Lambda}_2 \end{bmatrix}}_{\hat{K}} p = Q^T f(t) \tag{57}$$

which is, of course, equivalent to Eq. (53) for the K and D matrices given in Eqs. (55) and (56). Equation (57) can be rewritten as

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0.05 \\ 0 & 0 \end{bmatrix}}_{\hat{D}_1} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\hat{K}_1} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = g_1(t) \tag{58}$$

and

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 0.02 & 0 \\ 0 & 0.04 \end{bmatrix} \begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = g_2(t) \tag{59}$$

where $Q^T f(t) = [g_1^T(t), g_2^T(t)]^T$.

Equations (58) and (59) represent the decomposition of the system given by Eqs. (53), (55), and (56) into two independent uncoupled two degrees-of-freedom subsystems. Since the subsystem in Eq. (59) is diagonal, it can alternatively be thought of as representing two independent single-degree-of-freedom subsystems.

We further note that the matrix $\hat{D}_1$ in Eq. (58) can be split into its symmetric and skew-symmetric parts as

$$\hat{D}_1 = \hat{S}_1 + \hat{G}_1 = \begin{bmatrix} 0 & 0.025 \\ 0.025 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0.025 \\ -0.025 & 0 \end{bmatrix}$$

Since Eq. (58) represents a two degrees-of-freedom system in which $\hat{S}_1$ and the stiffness matrix $\hat{K}_1 = I_2$ commute, the quasi-diagonalization of this subsystem is guaranteed by *Lemma 3* through the use of an orthogonal transformation, which is the case.

Consider the subsystem in Eq. (58) now. We note that $[K, D] = 0$ implies $[\hat{D}_1, \hat{K}_1] = 0$. We can then ask the following question: since this commutation requirement (see *Remark 5*) is satisfied by this subsystem, can we then apply the result in Ref. [4] to it and ensure its diagonalization? The answer is, this subsystem cannot be diagonalized since the matrix $\hat{D}_1$ is defective. The reason why the guarantee in Ref. [4] fails is that the result there subsumes that $\hat{D}_1$ is non-defective. Alternatively, we can consider the system described by Eq. (57) in which the matrix $\hat{K}$ is symmetric; therefore, the equivalence transformation that diagonalizes $\hat{K}$ reduces to an orthogonal transformation. Again, $[\hat{D}, \hat{K}] = 0$. Recall that the matrix $\hat{D}$ has an eigenvalue equal to 0 with (algebraic) multiplicity two, and we find that corresponding to this multiple eigenvalue it has only one eigenvector. Hence, $\hat{D}$ is defective, and the system described by Eqs. (53), (55), and (56) cannot be diagonalized.

More generally, we see from *Remark 6* that when the multiplicities of the eigenvalues of K are at most two, then aiming for complete diagonalization, which gives uncoupled independent single-degree-of-freedom subsystems, appears to be perhaps too restrictive in trying to uncouple an MDOF system when $[K, D] = 0$; quasi-diagonalization relaxes this restriction and is guaranteed to provide uncoupled independent subsystems with at most *two* degrees-of-freedom because the necessary and sufficient conditions for this are automatically satisfied (see *Lemma 3*). ∎

Result 4. Let $S = S^T$, $G = -G^T$, $K = K^T$, and $\text{Rank}(G) = 2m \le n$. If all the eigenvalues of the matrix S are distinct, then there exists a real linear orthogonal change of coordinates such that the system shown in Eq. (7) transforms to the form given in Eqs. (33)–(36) if and only if the following conditions hold:

$$[S, G^2] = 0, \; [S, GSG] = 0, \; [K, S] = 0 \tag{60}$$

Proof. Interchange the n by n symmetric matrices K and S in *Result 3*. ∎

Remark 7. By interchanging K and S in *Remark 4* it follows that when all the eigenvalues of S are distinct and all the non-zero eigenvalues of G are distinct, the necessary and sufficient conditions reduce to $[S, G^2] = 0$ and $[K, S] = 0$. ∎

Given the three matrices S, G, and K the likelihood of their simultaneous quasi-diagonalization by a real orthogonal matrix Q increases as the number of independent commutation conditions that the three matrices are required to satisfy reduces, since each additional condition imposes additional constraints on the matrices. *Result 1* says that the number of the necessary and sufficient conditions needed for simultaneous quasi-diagonalization of S, G, and K, is seven. Our motivation in providing *Results 2–4* has been to reduce the number of these necessary and sufficient conditions for simultaneous quasi-diagonalization from a total of seven to just three by using information about the eigenvalues of the three matrices that often arise in engineering applications. Thinking along these lines, one is led to question whether one can further reduce the number of commutation relations required for simultaneous quasi-diagonalization by considering specific forms for the matrices, instead of information about their eigenvalues. We consider this topic next.

4 Simultaneous Quasi-Diagonalization Through the Imposition of Structure on the Matrices S and K

Among the results obtained so far the one that requires only two commutation conditions, is given in *Corollary 1* and we take this as the starting point in our exploration. However, the corollary guarantees the simultaneous quasi-diagonalization of only the matrices K and G by an orthogonal matrix Q. Using this corollary, one would have two options in obtaining the simultaneous quasi-diagonalization of all three matrices S, G, and K.

First, we could consider the matrices K and G. Application of *Corollary 1* to these two matrices says that a real orthogonal matrix Q exists such that the K and G can be simultaneously quasi-diagonalized if and only if $[K, G^2] = 0$ and $[K, GKG] = 0$ (Eqs. (27)

and (28)) are satisfied; for the symmetric matrix S to be also simultaneously diagonalized by this matrix Q, we would then require it to have a specific structure (form).

Alternatively, one could consider the matrices S and G to which one could apply *Corollary 1*, which says that a real orthogonal matrix Q exists so that S and G can be simultaneously quasi-diagonalize if and only if

$$[S, G^2] = 0 \text{ or } SG^2 = G^2 S \tag{61}$$

and

$$[S, GSG] = 0 \text{ or } (SG)^2 = (GS)^2 \tag{62}$$

For this matrix Q to then also simultaneously diagonalize K, the symmetric matrix K would need to have a specific structure (form). In obtaining suitable structures (forms) for the matrices K and S in the two respective options, our aim, of course, is to posit structures (forms) that would be general enough to encompass as wide a set of symmetric matrices as possible. We begin our exploration with the first option, and then take up the second.

4.1 Simultaneous Quasi-Diagonalization of Matrices *K* and *G*.

We begin this subsection with the following three lemmas.

Lemma 4.

(1) *The matrices K^u, G^{2v}, $(GKG)^w$ are symmetric for all integers $u, v, w \geq 0$. The spectra of the first two matrices are given by*
(2) $\Lambda[K^u] = \{\lambda_1^u, \lambda_2^u, \ldots, \lambda_n^u\}$
(3) $\Lambda[G^{2v}] = (-1)^v\{\beta_1^{2v}, \beta_1^{2v}, \ldots, \beta_m^{2v}, \beta_m^{2v}, 0, \ldots, 0\}$
where $\lambda_i, i = 1, 2, \ldots, n$ are real numbers, and $\beta_i \neq 0$, $i = 1, 2, \ldots m$, are real numbers.

Proof. Since K is symmetric, K^u is symmetric. The matrix G^2 is also symmetric, since G is skew-symmetric. Similarly, the matrix GKG is symmetric, as is $(GKG)^w$. Hence, the result in (1) above.

Regarding the spectra, since K is symmetric its eigenvalues, $\lambda_i, i = 1, 2, \ldots, n$, are real. This proves (2) above. Furthermore, as stated before, the non-zero eigenvalues of the n by n skew-symmetric matrix G with rank $2m \leq n$ are purely imaginary and come in the complex conjugate pairs $\{\pm i\beta_j\}_{j=1}^m$; the remainder of the $(n-2m)$ eigenvalues of G is zero. Hence, the spectrum of G^2 is given by

$$\Lambda(G^2) = -\{\beta_1^2, \beta_1^2, \ldots, \beta_m^2, \beta_m^2, 0, \ldots, 0\}$$

from which the result in (3) above follows. ∎

Lemma 5. *If $KG^2 = G^2K$ and $(KG)^2 = (GK)^2$, then for all $j, k \geq 0$*

$$\text{(a)} \quad (GKG)^j G^{2k} = G^{2k}(GKG)^j \tag{63}$$

$$\text{(b)} \quad G^{2k}K^j = K^j G^{2k} \tag{64}$$

$$\text{(c)} \quad (GKG)^k K^j = K^j (GKG)^k \tag{65}$$

Proof. The proof is somewhat long and is given in the Appendix. ∎

Lemma 6. *If $KG^2 = G^2K$ and $(KG)^2 = (GK)^2$, then*

(1) *the n by n matrix $B_{uvw} := K^u G^{2v}(GKG)^w$ is symmetric for all integers $u, v, w \geq 0$.*

(2) $$K^u G^{2v}(GKG)^w = K^u (GKG)^w G^{2v} = G^{2v} K^u (GKG)^w = G^{2v}(GKG)^w K^u = (GKG)^w G^{2v} K^u = (GKG)^w K^u G^{2v}$$

Thus, the order of multiplication of the three factors in B_{uvw} does not matter; the subscripts of B can be taken in any order.

Proof

(1) Taking the transposition of the matrix B_{uvw} we get

$$\begin{aligned}[K^u G^{2v}(GKG)^w]^T &= [(GKG)^T]^w [(G^2)^T]^v [K^T]^u \\ &= (GKG)^w G^{2v} K^u = (GKG)^w K^u G^{2v} \\ &= K^u (GKG)^w G^{2v} = K^u G^{2v}(GKG)^w.\end{aligned} \tag{66}$$

In the third equality above we have used Eq. (64) from *Lemma 5*; in the fourth equality, Eq. (65); and in last, Eq. (63).i

(2) We prove the first four equalities; the rest can be proved in a similar manner using *Lemma 5*.

$K^u G^{2v}(GKG)^w = K^u (GKG)^w G^{2v}$ follows from Eq. (63),
$K^u G^{2v}(GKG)^w = G^{2v} K^u (GKG)^w$ follows from Eq. (64),
$G^{2v} K^u (GKG)^w = G^{2v}(GKG)^w K^u$ follows from Eq. (65), etc. ∎

Consider now a special form of the symmetric part S of the damping matrix D (see Eqs. (2) and (7)) given by

$$S = \sum_{w=0}^{n-1}\sum_{v=0}^{h-1}\sum_{u=0}^{n-1} a_{uvw} K^u G^{2v}(GKG)^w = \sum_{w=0}^{n-1}\sum_{v=0}^{h-1}\sum_{u=0}^{n-1} a_{uvw} B_{uvw} \tag{67}$$

where the integer h is the degree of the minimal polynomial of G^2, which equals the number of distinct eigenvalues of G^2. The coefficients a_{uvw} in the summation are any real numbers; recall (*Lemma 6*) that the subscript u refers to the index of K, v to the index of G^2, and w to the index of (GKG). *Lemma 5* shows that when Eqs. (27) and (28) are satisfied the matrix S in Eq. (67) is symmetric, since every term following the last equality that is summed is symmetric.

Remark 8. The series form given in Eq. (67) for the symmetric part of the damping matrix D is quite versatile since the (real) coefficients a_{uvw} are arbitrary. The form includes expressions for S like

$$\begin{aligned}S &= \sum_{u=0}^{n-1} a_u K^u, \\ S &= a_0 I + \sum_{u=1}^{n-1}\sum_{v=1}^{h-1} a_{uv} K^u G^{2v} \text{ and} \\ S &= a_0 I + \sum_{u=1}^{n-1}[a_u K^u + b_u (GKG)^u] + \sum_{v=1}^{h-1} c_v G^{2v}\end{aligned} \tag{68}$$

as well as simpler sums made up of a few terms, as in expressions like

$$S = b_0 I + b_1 K + b_2 K^2 G^4 + b_3 G^2 (GKG)^3 \tag{69}$$

$$S = b_0 I + b_1 K + b_2 KG^2 + b_3 G^2 (GKG) \tag{70}$$

$$S = b_1 K^2 + b_2 G^2 + b_3 (GKG)^3 \tag{71}$$

The subscripted lowercase letters in Eqs. (68)–(71) stand for arbitrary real numbers. ∎

Result 5. Consider the dynamical system described by Eq. (7), namely

$$\ddot{x} + \underbrace{(S+G)}_{D}\dot{x} + Kx = f(t), \; G \neq 0 \tag{72}$$

in which the symmetric matrix S is given in Eq. (67), the skew-symmetric matrix G has rank $2m \leq n$, and $K = K^T$. The matrices K and G can be quasi-diagonalized if and only if

$$KG^2 = G^2 K \tag{73}$$

and

$$(KG)^2 = (GK)^2 \tag{74}$$

so that the dynamical system (Eq. (72)) can be uncoupled into independent subsystems, each with at most two degrees-of-freedom.

The uncoupled equations in the principal coordinate p have the form

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + \Lambda p = Q^T f(t) \tag{75}$$

where

$$\Sigma = \sum_{w=0}^{n-1} \sum_{v=0}^{h-1} \sum_{u=0}^{n-1} a_{uvw} \sigma_{uvw}$$

$$\begin{aligned}\sigma_{uvw} &= (-1)^{v+w} \mathrm{diag}(\lambda_1^u \lambda_2^w \beta_1^{2(v+w)}, \lambda_1^w \lambda_2^u \beta_1^{2(v+w)}, \ldots \\ &\quad \ldots, \lambda_{2m-1}^u \lambda_{2m}^w \beta_m^{2(v+w)}, \lambda_{2m-1}^w \lambda_{2m}^u \beta_m^{2(v+w)}, 0 \ldots, 0),\ v \text{ and/or } w \neq 0 \\ &= \mathrm{diag}(\lambda_1^u, \lambda_2^u, \ldots, \lambda_n^u),\ v, w = 0\end{aligned} \tag{76}$$

$$\begin{aligned}\Gamma &= \mathrm{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right) \\ &:= \mathrm{diag}(\beta_1 J_2, \ldots, \beta_m J_2, 0, \ldots, 0)\end{aligned} \tag{77}$$

and

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{78}$$

Thus, the real orthogonal matrix Q simultaneously quasi-diagonalizes D and K.

Proof. Using *Corollary 1*, we know that when Eqs. (73) and (74) are satisfied, a real orthogonal matrix Q exists such that the matrices G and K are simultaneously quasi-diagonalized, i.e., $Q^T KQ = \Lambda$, and $Q^T GQ = \Gamma$(see Eqs. (23) and (24)). Using this orthogonal matrix Q and the coordinate transformation $x = Qp$, Eq. (8) becomes

$$\ddot{p} + Q^T SQ\dot{p} + \Gamma\dot{p} + \Lambda p = Q^T f(t) \tag{79}$$

where

$$Q^T SQ = \sum_{w=0}^{n-1} \sum_{v=0}^{h-1} \sum_{u=0}^{n-1} a_{uvw} Q^T B_{uvw} Q \tag{80}$$

The general term in the summation above can be expressed as

$$\begin{aligned}Q^T B_{uvw} Q &= Q^T K^u G^{2v} (GKG)^w Q \\ &= [Q^T K^u Q][Q^T G^{2v} Q][Q^T (GKG)^w Q] \\ &= [\Lambda]^u [\Gamma^2]^v [\Gamma\Lambda\Gamma]^w = \sigma_{uvw}\end{aligned} \tag{81}$$

in which σ_{uvw} is the diagonal matrix given in Eq. (76). The last equality follows since Λ, Γ^2, and $\Gamma\Lambda\Gamma$ are diagonal matrices given in Eqs. (23), (25), and (26). We therefore have

$$Q^T SQ = \sum_{w=0}^{n-1} \sum_{v=0}^{h-1} \sum_{u=0}^{n-1} a_{uvw} \Lambda^u \Gamma^{2v} (\Gamma\Lambda\Gamma)^w \tag{82}$$

which is a diagonal matrix. ■

Result 6. When the rank$(G) = 2m \le n$ and all the non-zero eigenvalues of G are distinct and the symmetric part of the damping matrix has the form given in Eq. (67), then there exists a coordinate change $x = Qp$ with $Q^T Q = I$ that uncouples Eq. (8) into m independent two degrees-of-freedom subsystems, and $(n-2m)$ single-degree-of-freedom subsystems if and only if

$$KG^2 = G^2 K \tag{83}$$

The uncoupling described in Eqs. (75)–(78) remains valid.

Proof. When the non-zero eigenvalues G are all distinct, only one necessary and sufficient condition is needed for the simultaneous quasi-diagonalization of K and G, since the first condition, namely $KG^2 = G^2 K$, implies the second, $(KG)^2 = (GK)^2$, as shown in Ref. [8]. Simultaneous diagonalization of S (Eq. (67)) by Q follows as in *Result 5*. ■

We note that when the non-zero eigenvalues of G are distinct the number of necessary and sufficient conditions for quasi-diagonalization of K and G reduce from two (Eqs. (73) and (74)) to just the condition given in Eq. (73) when S has the form given in Eq. (67).

Example 3. To illustrate *Result 6*, consider the simple system given by

$$\ddot{x} + D\dot{x} + Kx = f(t) \tag{84}$$

in which the stiffness matrix $K = \mathrm{diag}(k_1, k_1, k_2)$, $k_i \neq 0$, $i = 1, 2$, the damping matrix

$$D = \begin{bmatrix} \gamma k_1^2 & 0 & -c \\ 0 & \gamma k_1^2 & 0 \\ c & 0 & \gamma k_2^2 \end{bmatrix} \tag{85}$$

with both γ and c real and non-zero numbers, and

$$f(t) = [f_1(t), f_2(t), f_3(t)]^T$$

Evidently, the matrix D is not symmetric, and it does not commute with the stiffness matrix K whose eigenvalue k_1 has multiplicity two. The symmetric and the skew-symmetric parts of the matrix D, denoted by S and G, respectively, are then (see *Lemma 1*)

$$S = \gamma\, \mathrm{diag}(k_1^2, k_1^2, k_2^2) \tag{86}$$

$$G = \begin{bmatrix} 0 & 0 & -c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \tag{87}$$

Since the spectrum of G is given by $\Lambda(G) = \{ci, -ci, 0\}$, $i = \sqrt{-1}$, the non-zero eigenvalues of G are distinct. Furthermore, the symmetric part, $S = \gamma K^2$, (Eq. (86)) of the damping matrix, D, has the form described in Eq. (67). Also, $KG^2 = G^2 K = -\mathrm{diag}(k_1 c_2, 0, k_2 c_2)$. *Result 6* is then applicable, and we are guaranteed the existence of a real orthogonal matrix Q that will simultaneously quasi-diagonalize the matrices D and K (or S, G, and K). Since the rank of G is two we expect this three degrees-of-freedom system to uncouple into a two degrees-of-freedom system and a single-degree-of-freedom system.

The real orthogonal matrix that simultaneously quasi-diagonalizes K and G is just the permutation matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{88}$$

because $Q^T KQ = \mathrm{diag}(k_1, k_2, k_1)$ and $Q^T GQ = -\mathrm{diag}(cJ_2, 0)$.

Using the coordinate transformation $x = Qp$ in Eq. (84), and pre-multiplying it by Q^T then leads to

$$\ddot{p} + \begin{bmatrix} \gamma k_1^2 & -c & 0 \\ c & \gamma k_2^2 & 0 \\ 0 & 0 & \gamma k_1^2 \end{bmatrix} \dot{p} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_1 \end{bmatrix} p = \begin{bmatrix} f_1(t) \\ f_3(t) \\ f_2(t) \end{bmatrix} \tag{89}$$

We see from Eq. (89) that the dynamical system shown in Eq. (84) uncouples into the two independent subsystems

$$\ddot{p}_1 + \begin{bmatrix} \gamma k_1^2 & -c \\ c & \gamma k_2^2 \end{bmatrix} \dot{p}_1 + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} p_1 = \begin{bmatrix} f_1(t) \\ f_3(t) \end{bmatrix} \tag{90}$$

and

$$\ddot{p}_2 + \gamma k_1^2 \dot{p}_2 + k_1 p_2 = f_2(t) \tag{91}$$

The first subsystem (Eq. (90)) has 2DOF and the second is a damped potential single-degree-of-freedom subsystem. ■

Remark 9. As explained in Sec. 1, in standard structural analysis the damping matrix $\hat{D}$ is often taken to be symmetric and (as equally often) taken to be of the form $\hat{D} = \alpha I + \beta K + \gamma K^2$. One could use this idea to further extend *Example 3*, so that the symmetric part S of the (arbitrary) damping matrix D is

$$S = \alpha I + \beta K + \gamma K^2 \tag{92}$$

where α, β, and γ are real constants and

$$D = \alpha I_3 + \beta K + \gamma K^2 + G \tag{93}$$

The new matrix D in Eq. (93) (with G in Eq. (87)) is still in the form given in Eq. (67) and one is again guaranteed by *Result 6* that the system with this new damping matrix D in Eq. (93) can again be uncoupled to yield two independent subsystems one having two degrees-of-freedom, the other having a single-degree-of-freedom. Only the diagonal terms of the matrix that multiplies $\dot{p}$ in Eq. (89) are affected. Equation (89) now changes to

$$\ddot{p} + \begin{bmatrix} \alpha + \beta k_1 + \gamma k_1^2 & -c & 0 \\ c & \alpha + \beta k_2 + \gamma k_2^2 & 0 \\ 0 & 0 & \alpha + \beta k_1 + \gamma k_1^2 \end{bmatrix} \dot{p} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_1 \end{bmatrix} p = \begin{bmatrix} f_1(t) \\ f_3(t) \\ f_2(t) \end{bmatrix}$$

■

Example 4. We next consider a larger, six degrees-of-freedom dynamical system described by

$$\ddot{x} + D\dot{x} + Kx = f(t)$$

that has a stiffness matrix given by

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 & 0 \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_5 & 0 \\ 0 & 0 & 0 & -k_5 & k_5 + k_6 & -k_6 \\ 0 & 0 & 0 & 0 & -k_6 & k_6 \end{bmatrix}$$

with $k_1 = k_2 = 2000$, $k_3 = k_4 = 1700$, and $k_5 = k_6 = 1400$. Such tri-diagonal stiffness matrices arise commonly in the analysis of structural and mechanical systems. The damping matrix D is taken to be (for brevity, we display the numerical values only up to four decimal places)

$$D = \begin{bmatrix} 3.6531 & -1.8289 & 1.1301 & -1.2345 & 0.3368 & 0.5262 \\ -1.6690 & 3.3981 & -1.8707 & 1.0917 & -0.8276 & 0.9671 \\ -1.1127 & -1.3012 & 3.1720 & -1.2990 & 0.8609 & -0.0469 \\ 0.9945 & -0.9636 & -1.6276 & 2.7167 & -0.5553 & 0.0543 \\ -0.0207 & 0.7277 & -1.1839 & -1.7736 & 2.6008 & -0.8789 \\ -0.6999 & -0.9716 & 0.1629 & -0.1794 & -1.7427 & 1.3526 \end{bmatrix}$$

The spectrum of D is

$$\Lambda(D) = \{0.4745 \pm 0.9307i,\ 2.8129 \pm 1.2481i,\ 5.1594 \pm 1.8005i\}$$

It has three pairs of complex eigenvalues. Since it has complex eigenvalues, there can be no real coordinate transformation that can uncouple this dynamical system into independent single-degree-of-freedom systems. However, if the conditions stated in *Result 6* are satisfied, then we would be able to uncouple this system into independent subsystem, each with at most two degrees-of-freedom by using a real coordinate transformation.

Splitting the matrix D into its symmetric part, S, and its skew-symmetric part, G, we get

$$S = \begin{bmatrix} 3.6531 & -1.7489 & 0.0087 & -0.1200 & 0.1581 & -0.0869 \\ -1.7489 & 3.3981 & -1.5860 & 0.0640 & -0.0499 & -0.0023 \\ 0.0087 & -1.5860 & 3.1720 & -1.4633 & -0.1615 & 0.0580 \\ -0.1200 & 0.0640 & -1.4633 & 2.7167 & -1.1644 & -0.0625 \\ 0.1581 & -0.0499 & -0.1615 & -1.1644 & 2.6008 & -1.3108 \\ -0.0869 & -0.0023 & 0.0580 & -0.0625 & -1.3108 & 1.3526 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0 & -0.0800 & 1.1214 & -1.1145 & 0.1788 & 0.6130 \\ 0.0800 & 0 & -0.2848 & 1.0277 & -0.7777 & 0.9694 \\ -1.1214 & 0.2848 & 0 & 0.1643 & 1.0224 & -0.1049 \\ 1.1145 & -1.0277 & -0.1643 & 0 & 0.6092 & 0.1168 \\ -0.1788 & 0.7777 & -1.0224 & -0.6092 & 0 & 0.4319 \\ -0.6130 & -0.9694 & 0.1049 & -0.1168 & -0.4319 & 0 \end{bmatrix}$$

The spectrum of G is $\{\pm i, \pm 1.5i, \pm 2i\}$, showing that its eigenvalues are distinct, and hence *Result 6* could be used; by *Result 6* we need the condition given in Eq. (83) to be satisfied by the matrices K and G. A simple computation shows that this condition is satisfied. Furthermore, we find that the symmetric part, S, of the damping matrix D can be expressed as (see Eq. (70))

$$S = (0.01)[I_6 + 0.1K - (0.001)KG^2 - (0.001)G^2(GKG)]$$

which has the form given in Eq. (67) where only four terms of the series are present.

Result 6 is therefore applicable, and we are guaranteed that there exists a real orthogonal matrix Q capable of decoupling the system into independent subsystems, each having at most two degrees-of-freedom. Since G has three non-zero complex conjugate pairs of eigenvalues, we expect this six degrees-of-freedom system to uncouple into 3 two degrees-of-freedom subsystems.

Indeed, using the coordinate transformation $x = Qp$, where the real orthogonal matrix

$$Q = \begin{bmatrix} 0.1191 & -0.3226 & 0.4748 & -0.4387 & -0.4676 & -0.4951 \\ 0.2319 & -0.5125 & 0.4490 & -0.0747 & 0.2632 & 0.6380 \\ 0.3502 & -0.4877 & -0.1383 & 0.5144 & 0.3293 & -0.4973 \\ 0.4467 & -0.2268 & -0.5541 & -0.0037 & -0.5976 & 0.2915 \\ 0.5301 & 0.2234 & -0.2244 & 0.6231 & 0.4648 & -0.1201 \\ 0.5734 & 0.5423 & 0.4433 & 0.3861 & -0.1747 & 0.0325 \end{bmatrix} \quad (94)$$

we obtain the relation in Eq. (75) (which we repeat here for convenience)

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + \Lambda p = Q^T f(t) \quad (95)$$

in which

$$\Sigma = Q^T SQ = \text{diag}(0.1086, 0.8403, 1.9809, 3.6448, 4.2886, 6.0302)$$
$$\Gamma = Q^T GQ = \text{diag}(J_2, 1.5J_2, 2.0J_2)$$
$$\Lambda = Q^T KQ = \text{diag}(105.8087, 823.1451, 2108.6889, 3659.2502, 5125.8813, 6577.2255)$$

and

$$Q^T f(t) := [g_1^T(t), g_2^T(t), g_3^T(t)]^T$$

Inserting these matrices into Eq. (95) we see that this six degrees-of-freedom system uncouples, as expected, into three different independent two degrees-of-freedom subsystems given by

$$\ddot{p} + \begin{bmatrix} 0.1086 & 1 \\ -1 & 0.8403 \end{bmatrix}\dot{p} + \begin{bmatrix} 105.8087 & 0 \\ 0 & 823.1451 \end{bmatrix} p = g_1(t)$$

$$\ddot{p} + \begin{bmatrix} 1.9809 & 1.5 \\ -1.5 & 3.6448 \end{bmatrix}\dot{p} + \begin{bmatrix} 2108.6889 & 0 \\ 0 & 3659.2502 \end{bmatrix} p = g_2(t)$$

and

$$\ddot{p} + \begin{bmatrix} 4.2886 & 2 \\ -2 & 6.0302 \end{bmatrix}\dot{p} + \begin{bmatrix} 5125.8813 & 0 \\ 0 & 6577.2255 \end{bmatrix} p = g_3(t)$$

As stated in *Result 2*, the number of two degrees-of-freedom subsystems is $m = 3$ where $\text{rank}(G) = 2m$. ■

When K has distinct eigenvalues then we have the following result.

Result 7. Consider the damping matrix $D = S + G$ where S is the symmetric part of D, and G the skew-symmetric part. Let S have the form given in Eq. (67). When all the eigenvalues of K (and/or GKG) are distinct, the conditions given in Eqs. (73) and (74) are necessary and sufficient for the existence of a real coordinate transformation $p = Qx(Q^TQ = I)$ so that K and G can be simultaneously quasi-diagonalized and the dynamical system

$$\ddot{x} + D\dot{x} + Kx = f(t)$$

is decomposed into independent uncoupled subsystems with at most two degrees-of-freedom. Furthermore, every matrix D that permits such a quasi-diagonalization of K and G must have a symmetric part S that can always be expressed in the form (see Eq. (68))

$$S = a_0 I + \sum_{u=1}^{n-1}(a_u K^u + b_u (GKG)^u) + \sum_{v=1}^{h-1} c_v G^{2v} \quad (96)$$

where a_0, a_u, b_u, and c_v are real numbers.

Proof. We note that by *Result 5*, Eqs. (73) and (74) are necessary and sufficient for a real orthogonal matrix Q to exist such that in the equation (see Eq. (8))

$$\ddot{p} + Q^T SQ\dot{p} + Q^T GQ\dot{p} + Q^T KQp = Q^T f(t)$$
$$Q^T GQ = \text{diag}(\beta_1 J_2, \beta_2 J_2, \ldots, \beta_m J_2, 0, \ldots 0),$$
$$\text{and } Q^T KQ = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Hence the dynamical system can be quasi-diagonalized. Reference [10] shows that when Q quasi-diagonalizes K and G and the eigenvalues of K (and/or GKG) are distinct, any symmetric matrix S that Q diagonalizes can always be expressed in the simpler form given in Eq. (96) (see the last relation in Eq. (68), *Remark 8*).■

Corollary 5. *Consider the damping matrix $D = S + G$ where S is the symmetric part of D and G the skew-symmetric part. Let S have the form given in Eq. (67). When*

(1) *all the eigenvalues of K are distinct,*
(2) *all the non-zero eigenvalues of G are also distinct,*

and

(3) *Equation (73) is satisfied,*

the existence of a real coordinate transformation $p = Qx(Q^TQ = I)$ is guaranteed so that the equation

$$\ddot{x} + D\dot{x} + Kx = f(t)$$

is decomposed into independent uncoupled subsystems with at most two degrees-of-freedom. Furthermore, every matrix D that permits such a quasi-diagonalization must have a symmetric part S that can be expressed in the form given in Eq. (96).

Proof. Result 6 shows that when the non-zero eigenvalues of G are all distinct, only Eq. (83) is necessary and sufficient to simultaneously quasi-diagonalize K and G. The result then follows upon application of *Result 7*. ■

Remark 10. When the eigenvalues of K are distinct, the orthonormal eigenvectors of K are unique, and the eigenspace of each eigenvalue has dimension one. Hence, finding the columns of the matrix Q that simultaneously quasi-diagonalizes K and G when Eqs. (73)

and (74) are satisfied is much simpler, since they must be the eigenvectors of K. This is indeed how Q (Eq. (94)) was obtained in *Example 4*. When, in addition, the non-zero eigenvalues of G are distinct one only needs to check if Eq. (73) (also Eq. (83)) is satisfied to guarantee the existence of a real coordinate transformation $x = Qp$ that will simultaneously quasi-diagonalize K and G. ■

Remark 11. The matrix Q in *Result 6* can be easily obtained as follows. Let $\hat{Q}$ be an orthogonal matrix which diagonalizes the symmetric matrix G^2, so that $\hat{Q}^T G^2 \hat{Q} = -\text{diag}(\beta_1^2 I_2, \ldots, \beta_m^2 I_2, 0_{n-2m})$, where I_2 is the two-dimensional identity matrix and 0_{n-2m} is the $(n-2m)$-dimensional zero matrix, β_i are real and non-zero; $\beta_i \neq \beta_j$, for $i \neq j$. Since G^2 commutes with K and since $\beta_i \neq \beta_j$ for $i \neq j$, we must have $\hat{Q}^T K \hat{Q} = \text{diag}(K_1, \ldots, K_m, K_{n-2m})$ where $K_i (i = 1, 2, \ldots m)$ are each two by two real symmetric matrices, and K_{n-2m} is an $(n-2m)$ by $(n-2m)$ real symmetric matrix. Furthermore, since G^2 also commutes with G, we must have $\hat{Q}^T G \hat{Q} = \text{diag}(G_1, \ldots, G_m, 0_{n-2m})$, where $G_i (i = 1, 2, \ldots m)$ are each two by two skew-symmetric matrices. It follows that the orthogonal matrix Q that simultaneously quasi-diagonalizes K and G has the form $Q = \hat{Q} diag(\bar{Q}_1, \ldots, \bar{Q}_m, \bar{Q}_{n-2m})$, where $\bar{Q}_i (i = 1, \ldots, m)$ is an orthogonal two by two matrix that diagonalizes K_i, and $\bar{Q}_{n-2m}$ is an $(n-2m)$ by $(n-2m)$ orthogonal matrix that diagonalizes K_{n-2m}. ■

Remark 12. When both K and G have multiple eigenvalues the determination of a real orthogonal transformation Q gets more involved. As explained earlier, systems with such multiple eigenvalues are rare in real-life applications, and a general procedure for determining Q is described in detail in the Appendix of Ref. [8]. ■

4.2 Simultaneous Quasi-Diagonalization of Matrices *S* and *G*.

As discussed before, the uncoupling of a damped potential system can also be accomplished, through the simultaneous quasi-diagonalization of the symmetric matrix S and the skew-symmetric matrix G which form the components of the damping matrix D if the proper conditions are satisfied. In this section, we explore this avenue and give the conditions.

We note that *Corollary 1* applies to two matrices, one symmetric, and the other skew-symmetric. Instead of taking these two matrices to be K and G (as we did in *Result 5*), we now take the two matrices to be S and G, and apply *Corollary 1*. This simply involves the interchange of the symbols K and S. Interchanging them in Eq. (67), we consider the matrix K to have the form

$$K = \sum_{w=0}^{n-1} \sum_{v=0}^{h-1} \sum_{u=0}^{n-1} c_{uvw} S^u G^{2v} (GSG)^w = \sum_{w=0}^{n-1} \sum_{v=0}^{h-1} \sum_{u=0}^{n-1} c_{uvw} C_{uvw} \quad (97)$$

where c_{uvw} are arbitrary real numbers. Provided $SG^2 = G^2 S$ and $(SG)^2 = (GS)^2$, *Lemmas 2–5* are valid when K is replaced by the symmetric matrix S in them; symmetry of each term in the summation in Eq. (97) therefore follows. We then obtain a result analogous to *Result 5*.

Remark 13. Just like S in Eq. (67), the form of K in Eq. (97) is quite versatile. Expressions for K analogous to those for S (see *Remark 8*) can be obtained by interchanging K and S in Eqs. (68)–(71) in *Remark 8*. ■

Result 8. Consider the dynamical system described by Eq. (7), namely

$$\ddot{x} + \underbrace{(S + G)}_{D} \dot{x} + Kx = f(t) \quad (98)$$

in which the symmetric matrix K has the form given in Eq. (97), the skew-symmetric matrix $G \neq 0$ has rank $2m \leq n$, and $S = S^T$. The matrices S and G can be simultaneously quasi-diagonalized by a real orthogonal matrix Q if and only if

$$SG^2 = G^2 S \quad (99)$$

and

$$(SG)^2 = (GS)^2 \quad (100)$$

so that the dynamical system(Eq. (98)) can be uncoupled into independent subsystems, each with at most two degrees-of-freedom.

The uncoupled equations in the principal coordinate p defined by the real transformation $x = Qp$ have the form

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + \Lambda p = Q^T f(t) \quad (101)$$

where

$$\Sigma = diag(\sigma_1, \sigma_1, \ldots, \sigma_n) \quad (102)$$

$$\Gamma = \text{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right) \\ := \text{diag}(\beta_1 J_2, \ldots, \beta_m J_2, 0, \ldots 0) \quad (103)$$

$$\Lambda = \sum_{w=0}^{n-1} \sum_{v=0}^{h-1} \sum_{u=0}^{n-1} c_{uvw} \zeta_{uvw} \quad (104)$$

and

$$\zeta_{uvw} = (-1)^{v+w} \text{diag}(\sigma_1^u \sigma_2^w \beta_1^{2(v+w)}, \sigma_1^w \sigma_2^u \beta_1^{2(v+w)}, \ldots \\ \ldots, \sigma_{2m-1}^u \sigma_{2m}^w \beta_m^{2(v+w)}, \sigma_{2m-1}^w \sigma_{2m}^u \beta_m^{2(v+w)}, 0 \ldots, 0),\ v \text{ and/or } w \neq 0 \\ = \text{diag}(\sigma_1^u, \sigma_2^u, \ldots, \sigma_n^u),\ v, w = 0. \quad (105)$$

Thus, the real orthogonal matrix Q simultaneously quasi-diagonalizes D and K.

Proof. Using *Corollary 1*, we observe that if and only if Eqs. (99) and (100) are satisfied, a real orthogonal matrix Q exists such that the matrices G and S can be simultaneously quasi-diagonalized, i.e., $Q^T SQ = \Sigma$, and $Q^T GQ = \Gamma$. Using this orthogonal matrix Q and the coordinate transformation $x = Qp$, Eq. (8) becomes

$$\ddot{p} + \Sigma\dot{p} + \Gamma\dot{p} + Q^T KQp = Q^T f(t)$$

where

$$Q^T KQ = \sum_{u=0}^{n-1} \sum_{v=0}^{h-1} \sum_{w=0}^{n-1} c_{uvw} Q^T C_{uvw} Q$$

The general term in the summation above can be expressed as

$$\begin{aligned} Q^T C_{uvw} Q &= Q^T S^u G^{2v} (GSG)^w Q \\ &= [Q^T S^u Q][Q^T G^{2v} Q][Q^T (GSG)^w Q] \\ &= [\Sigma]^u [\Gamma^2]^v [\Gamma\Sigma\Gamma]^w = \xi_{uvw} \end{aligned}$$

in which ξ_{uvw} is the diagonal matrix given in Eq. (105), and

$$Q^T KQ = \sum_{u=0}^{n-1} \sum_{v=0}^{h-1} \sum_{w=0}^{n-1} c_{uvw} \Sigma^u \Gamma^{2v} (\Gamma\Sigma\Gamma)^w$$

which is a diagonal matrix. ■

As discussed before, and illustrated in the proof above, new results analogous to all the results obtained in Sec. 4.1 can be obtained by simply interchanging the symbols K and S and noting the changes in the expressions for Σ and Λ in Eqs. (102) and (104). For example, the analog of *Corollary 5* is as follows.

Corollary 6. *Consider the damping matrix $D = S + G$ where S is the symmetric part of D and G the skew-symmetric part. Let K have the form given in Eq. (97). When*

(1) *all the eigenvalues of S are distinct,*
(2) *all the non-zero eigenvalues of G are also distinct,*
and
(3) *Equation (99) is satisfied,*

the existence of a real coordinate transformation $p = Qx(Q^TQ = I)$ is guaranteed so that the equation

$$\ddot{x} + \underbrace{(S+G)}_{D}\dot{x} + Kx = f(t)$$

is decomposed into independent uncoupled subsystems with at most two degrees-of-freedom. Furthermore, every matrix K that permits such a quasi-diagonalization must be expressed in the form given in

$$K = a_0 I + \sum_{u=1}^{n-1}(a_u S^u + b_u (GSG)^u) + \sum_{v=1}^{h-1} c_v G^{2v} \tag{106}$$

where a_0, a_u, b_u, and c_v are real numbers. ■

Example 5. We illustrate the application of *Corollary 6* to a damped MDOF gyroscopic potential dynamical system (in which G is the gyroscopic matrix, K is the potential (stiffness) matrix, and S is the symmetric damping matrix) described by the equation

$$\ddot{x} + S\dot{x} + G\dot{x} + Kx = f(t)$$

where

$$S = \begin{bmatrix} 0.0424 & 0.0433 & 0.0159 & 0.0083 & 0.0327 \\ 0.0433 & 0.0732 & 0.0111 & 0.0073 & 0.0200 \\ 0.0159 & 0.0111 & 0.0511 & 0.0276 & 0.0308 \\ 0.0083 & 0.0073 & 0.0276 & 0.0827 & 0.0128 \\ 0.0327 & 0.0200 & 0.0308 & 0.0128 & 0.0405 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & -0.1075 & -0.4986 & -0.0168 & 0.4495 \\ 0.1075 & 0 & 0.1310 & -0.1967 & -0.3526 \\ 0.4986 & -0.1310 & 0 & -0.0332 & -0.1874 \\ 0.0168 & 0.1967 & 0.0332 & 0 & 0.1537 \\ -0.4495 & 0.3526 & 0.1874 & -0.1537 & 0 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 252.4193 & -88.5826 & -26.8433 & 19.8665 & -48.0309 \\ -88.5826 & 132.4679 & 68.6458 & -20.3464 & 13.2369 \\ -26.8433 & 68.6458 & 176.5082 & -9.3011 & -117.2071 \\ 19.8665 & -20.3464 & -9.3011 & 73.1958 & 40.4367 \\ -48.0309 & 13.2369 & -117.2071 & 40.4367 & 212.7946 \end{bmatrix}$$

The spectrum of G is $\{\pm 0.25i, \pm 0.8i, 0\}$, and we see that the dynamical system cannot be diagonalized by any real transformation. The non-zero eigenvalues of G are seen to be distinct. The eigenvalues of S are also distinct and Eq. (99) is satisfied by the matrices S and G. Furthermore, the matrix K can be expressed as

$$K = 500[S - G^2 - (GSG)^2] \tag{107}$$

which is in the form given in Eq. (106).

Hence, by *Corollary 6*, an orthogonal matrix Q exists that simultaneously quasi-diagonalizes these two matrices. Indeed, it is given by

$$Q = \begin{bmatrix} -0.4629 & -0.3040 & -0.7004 & -0.4495 & 0.0251 \\ -0.5204 & -0.5069 & 0.2847 & 0.4086 & -0.4735 \\ -0.4096 & 0.2778 & -0.2328 & 0.6275 & 0.5542 \\ -0.4083 & 0.7561 & 0.0290 & -0.1630 & -0.4840 \\ -0.4248 & -0.0423 & 0.6110 & -0.4589 & 0.4835 \end{bmatrix}$$

and it leads to

$$Q^TDQ = \begin{bmatrix} 0.1426 & 0.2500 & 0 & 0 & 0 \\ -0.2500 & 0.0839 & 0 & 0 & 0 \\ 0 & 0 & 0.0013 & 0.8000 & 0 \\ 0 & 0 & -0.8000 & 0.0172 & 0 \\ 0 & 0 & 0 & 0 & 0.0450 \end{bmatrix} \tag{108}$$

The matrix Q simultaneously diagonalizes K, and we have

$$Q^TKQ = \mathrm{diag}(102.5364,\ 73.1681,\ 320.5885,\ 328.5816,\ 22.5112) \tag{109}$$

Using the coordinate transformation $x = Qp$, in Eq. (98) and multiplying it by Q^T from the left, upon noting Eqs. (108) and (109) we then get the relation

$$\ddot{p} + \underbrace{\begin{bmatrix} 0.1426 & 0.2500 & 0 & 0 & 0 \\ -0.2500 & 0.0839 & 0 & 0 & 0 \\ 0 & 0 & 0.0013 & 0.8000 & 0 \\ 0 & 0 & -0.8000 & 0.0172 & 0 \\ 0 & 0 & 0 & 0 & 0.0450 \end{bmatrix}}_{\Sigma+\Gamma}\dot{p} + \begin{bmatrix} 102.5364 & 0 & 0 & 0 & 0 \\ 0 & 73.1681 & 0 & 0 & 0 \\ 0 & 0 & 320.5885 & 0 & 0 \\ 0 & 0 & 0 & 328.5816 & 0 \\ 0 & 0 & 0 & 0 & 22.5112 \end{bmatrix}p = Q^Tf(t)$$

This gives the following three uncoupled subsystems:

$$\ddot{p}_1 + \begin{bmatrix} 0.1426 & 0.2500 \\ -0.2500 & 0.0839 \end{bmatrix}\dot{p}_1 + \begin{bmatrix} 102.5364 & 0 \\ 0 & 73.1681 \end{bmatrix}p_1 = g_1(t)$$

$$\ddot{p}_2 + \begin{bmatrix} 0.0013 & 0.8000 \\ -0.8000 & 0.0172 \end{bmatrix}\dot{p}_2 + \begin{bmatrix} 320.5885 & 0 \\ 0 & 328.5816 \end{bmatrix}p_2 = g_2(t)$$

and

$$\ddot{p}_3 + 0.045\,\dot{p}_3 + 22.5112 p_3 = g_3(t)$$

where $Q^Tf(t) = [g_1^T(t),\ g_2^T(t),\ g_3(t)]^T$.

As seen, two of the subsystems have two degrees-of-freedom and one is a single-degree-of-freedom system. The corollary further guarantees that K given in Eq. (107) can be expressed in the form given in Eq. (106), which, in this case, it already is. ■

5 Conclusions

This paper deals with the uncoupling of damped linear MDOF potential systems with arbitrary damping matrices into smaller-dimensional subsystems with at most two degrees-of-freedom through the use of a simple *real* linear coordinate transformation that uses an *orthogonal* matrix. It is well-known that two symmetric matrices can be simultaneously diagonalized by a real orthogonal transformation if and only if they commute. This important result in linear algebra provides a simple linear coordinate transformation that uses an orthogonal matrix and that permits the decomposition of a linear MDOF damped potential system—with a positive definite mass matrix, and symmetric stiffness and damping matrices that commute—into a set of independent, uncoupled single-degree-of-freedom subsystems. The damped system has normal modes of vibration, and this permits its behavior to be easily understood and robustly computed [2].

Since the publication of this important result about 60 years ago, models of engineered and naturally occurring systems employed in academic research and engineering practice often restrict the damping matrices in linear MDOF potential systems to be symmetric and to commute with the symmetric potential (stiffness) matrix. These restrictions (assumptions) on the damping matrix are placed not least because they enable the uncoupling of such systems through the use of an orthogonal coordinate transformation into single-degree-of-freedom independent subsystems, whose vibratory behavior is well understood. However, in complex vibratory structural and mechanical systems the sources of damping are often difficult to assess/identify. Also, experimental measurements of damping (and its linearized approximations) used in the modeling of linear MDOF systems often show that the damping matrix may not commute with the stiffness matrix and/or may not be symmetric.

To the best of the authors' knowledge, a continuation of the fruitful line of exploration introduced in Ref. [2] that uses an orthogonal matrix transformation of the coordinates has not been undertaken when the two restrictions stated above on the damping matrix are disposed of. Besides catering to robust computational procedures, orthogonal transformations are simple to understand since they physically represent rotations and reflections. This paper addresses linear MDOF potential systems in which the damping matrices can be arbitrary; it simultaneously handles MDOF gyroscopic potential systems with symmetric damping matrices, which are their duals. We seek linear coordinate transformations that use orthogonal matrices to maximally uncouple such MDOF systems. The uncoupling leads to improved physical insights into their dynamical behavior, and the orthogonal matrices employed in the coordinate transformations provide robust methods for the computation of their responses to external excitations.

The main results can be summarized as follows:

(1) A new central result in linear algebra that gives the n&s conditions for two n by n symmetric matrices and one skew-symmetric matrix to be simultaneously quasi-diagonalized by a real orthogonal congruence is proved in Sec. 2. A total of ten n&s conditions are found, which are then reduced to seven n&s conditions. Reducing the number of n&s conditions while keeping the set of matrices as wide as possible, so that the results are of practical value and can be applied to engineered systems as well as those found in nature, is one of the underlying threads in the paper.

(2) The existing literature to date places damped MDOF potential systems in a different dynamical category from damped gyroscopic systems. This is understandably so, because from a physical standpoint, the principal forces that engender their dynamical behavior—potential forces and gyroscopic forces—are widely different in their origin and character [6].

The paper shows, however, that from a mathematical standpoint, a given linear MDOF potential system with stiffness matrix K and an arbitrary damping matrix $D = S + G$, where S is symmetric and $G \neq 0$ is skew-symmetric, is identical in its dynamical behavior to an MDOF gyroscopic potential system with the stiffness matrix K, gyroscopic matrix G, and symmetric damping matrix S. Because these two physically dissimilar systems share a common equation of motion, we consider them as being duals (or reflections) of each other, and the two disparate categories to which they belong are thus brought together under a unified conceptual framework.

For simplicity and brevity, all the results in this paper (as also in the discussion below) are introduced mainly using the notion of a damped MDOF potential system with an arbitrary damping matrix. The dynamical behavior of its dual to any external excitation being identical, all the results herein are also therefore equally applicable to damped MDOF gyroscopic potential systems with symmetric damping matrices.

(3) Taking the symmetric matrices to be the matrices S and K and the skew-symmetric matrix to be G, in Sec. 3, the central results described in item (1) above are used to give the n&s conditions for a damped MDOF potential system (with an arbitrary damping matrix $D \neq 0$) so that it is uncoupled by a real linear coordinate transformation using an orthogonal matrix. The uncoupling is maximal in the sense that it leads to a decomposition of the MDOF system into at most two degrees-of-freedom, independent subsystems. As mentioned before, such an uncoupling lends itself to considerably greater physical insights into the vibratory behavior of such MDOF potential systems and, in addition, results in robust computational schemes for quantitatively determining their response to external forces.

(4) Noting that the ten n&s conditions—though winnowed down to seven commutation conditions—still constitute significant restrictions on the matrices that describe a linear MDOF potential system with an arbitrary damping matrix, they are further reduced to three. Several results are obtained that show that these three n&s conditions can be used when modeling physical systems that commonly arise in civil, aerospace, and mechanical engineering.

(5) To reduce the number of n&s conditions more generally, from seven down to just two, we posit (consider) a general form for one of the two symmetric matrices (see item (3) above), say, the symmetric part S of the damping matrix (or the stiffness matrix K). This general form that is posited is chosen to be versatile in the sense that it encompasses a wide variety of symmetric matrices. The form automatically satisfies the remaining five n&s conditions, leaving only two n&s conditions to be imposed on a damped MDOF potential system to guarantee its maximal uncoupling. When the matrix S has the posited form, several results related to the maximal uncoupling of such systems are presented, along with numerical illustrative examples.

Further, when the gyroscopic part of the damping matrix in our damped MDOF potential system has distinct non-zero eigenvalues, it is shown that only a single n&s condition is required to maximally uncouple the system whose S matrix has the posited form. Such systems are often encountered in engineering practice and in nature. In addition, if the stiffness matrix has distinct eigenvalues then it is shown that the symmetric part of the damping matrix *must* be expressible in a simple form, which is explicitly obtained herein, and which is a subset of the general form posited.

As mentioned earlier, we can alternatively obtain the results for the dual MDOF gyroscopic potential system that has a symmetric damping matrix S. For example, when the stiffness matrix K is in the general posited form provided (Eq. (97)), two n&s conditions (that we provide explicitly) are required to be satisfied for uncoupling the MDOF system into at most two degrees-of-freedom, independent subsystems. If, in addition, the non-zero eigenvalues of G are distinct, only one n&s condition guarantees such an uncoupling. Furthermore, when S has distinct eigenvalues, the stiffness matrix K must be expressible in a simple form (Eq. (106)), which is a subset of the general posited form.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

Appendix

Lemma 5. *When*

$$KG^2 = G^2K \tag{A1}$$

and

$$(KG)^2 = (GK)^2 \tag{A2}$$

then

$K^j, G^{2k}, (GKG)^r$, *integer* $j, k, r \geq 0$, *commute pairwise. In other words, for* $j, k \geq 0$

$$\text{(a)} \quad G^{2k}(GKG)^j = (GKG)^j G^{2k} \tag{A3}$$

$$\text{(b)} \quad K^j G^{2k} = G^{2k} K^j \tag{A4}$$

$$\text{(c)} \quad K^j (GKG)^k = (GKG)^k K^j \tag{A5}$$

$$\text{(d)} \quad G(GKG)^j = G^{2j} K^j G = K^j G^{2j} G \tag{A6}$$

Proof.

(a) Equation (A3) is obviously true for $k = 0$ and/or $j = 0$. We first show that

$$G^{2k}(GKG) = (GKG)G^{2k}, \; k \geq 1 \tag{A7}$$

We begin by showing that this is true for $k = 1$, because

$$G^2(GKG) = G(G^2K)G = G(KG^2)G = (GKG)G^2 \tag{A8}$$

in which the second equality follows from Eq. (A1). We prove the relation in Eq. (A7) by induction. We assume that Eq. (A7) is true for some $k = (l-1) > 0$, so that

$$G^{2l-2}(GKG) = (GKG)G^{2l-2} \tag{A9}$$

Then,

$$\begin{aligned} G^{2l}(GKG) &= G^2G^{2l-2}(GKG) = G^2(GKG)G^{2l-2} \\ &= (GKG)G^2G^{2l-2} = (GKG)G^{2l} \end{aligned} \tag{A10}$$

where we have used Eq. (A9) in the second equality and Eq. (A8) in the third. This completes our inductive proof of Eq. (A7).

We next show that Eq. (A3) is true. From Eq. (A7), we see that Eq. (A3) is true for $j = 1$. Assume that it is true for $j = l - 1 > 0$, i.e.,

$$G^{2k}(GKG)^{l-1} = (GKG)^{l-1}G^{2k} \tag{A11}$$

Then,

$$\begin{aligned} G^{2k}(GKG)^l &= G^{2k}(GKG)^{l-1}(GKG) = (GKG)^{l-1}G^{2k}(GKG) \\ &= (GKG)^{l-1}(GKG)G^{2k} = (GKG)^l G^{2k} \end{aligned} \tag{A12}$$

where we have used Eq. (A11) in the second equality and Eq. (A7) in the third equality.

(b) Equation (A4) is obviously true for $k = 0$ and/or $j = 0$. We begin by showing that $K^jG^2 = G^2K^j, j > 1$. We note that when $j = 1$, this relation calls for the commutation of the matrices K and G^2, which is true because of Eq. (A1). We now assume that this relation is true for some integer $j = (l-1) > 0$, so that

$$K^{l-1}G^2 = G^2K^{l-1} \tag{A13}$$

Then,

$$K^lG^2 = K(K^{l-1}G^2) = K(G^2K^{l-1}) = G^2KK^{l-1} = G^2K^l$$

thereby completing the induction argument, which shows that

$$K^jG^2 = G^2K^j, \; j > 1 \tag{A14}$$

We now fix a certain arbitrary value of $j = j* > 1$ and consider the relation

$$K^{j*}G^{2k} = G^{2k}K^{j*}, \; k > 1 \tag{A15}$$

which, by Eq. (A14), is true for $k = 1$. We assume next that Eq. (A15) is true for some integer $k = (l-1) > 0$, so that $K^{j*}G^{2l-2} = G^{2l-2}K^{j*}, k > 1$. Now we find that

$$K^{j*}G^{2l} = K^{j*}G^{2l-2}G^2 = G^{2l-2}K^{j*}G^2 = G^{2l-2}G^2K^{j*} = G^{2l}K^{j*} \tag{A16}$$

where we have used the equality in Eq. (A14), which is valid for $j = j^*$. Since $j^* > 1$ is arbitrary, this completes the proof by induction of Eq. (A4).

(c) Equation (A5) is obviously true for $k = 0$ and/or $j = 0$. We begin by showing that

$$K^j(GKG) = (GKG)K^j \tag{A17}$$

When $j = 1$, Eq. (A17) calls for the commutation of K and GKG, which is true because of Eq. (A2). We now assume that Eq. (A17) is true for some integer $j = (l-1) > 0$ so that

$$K^{l-1}(GKG) = (GKG)K^{l-1} \tag{A18}$$

Then,

$$\begin{aligned} K^l(GKG) &= KK^{l-1}(GKG) = K(GKG)K^{l-1} \\ &= (GKGK)K^{l-1} = (GKG)K^l \end{aligned} \tag{A19}$$

which shows that the Eq. (A17) in true. We now continue to show that Eq. (A5) is true. From Eq. (A17), we see that it is true for $k = 1$. We assume that it is true for $k = (l-1) > 0$, i.e.,

$$K^j(GKG)^{l-1} = (GKG)^{l-1}K^j \tag{A20}$$

Then,

$$\begin{aligned} K^j(GKG)^l &= K^j(GKG)^{l-1}(GKG) = (GKG)^{l-1}K^j(GKG) \\ &= (GKG)^{l-1}(GKG)K^j = (GKG)^lK^j \end{aligned} \tag{A21}$$

where we have used Eq. (A20) in the second equality and Eq. (A17) in the third.

(d) Equation (A6) is satisfied for $j = 1$ since $G(GKG) = G^2KG$. We assume that it is satisfied for $j = l - 1 > 0$, and show that it then is satisfied for $j = l$. Using Eq. (A4), we get

$$\begin{aligned} G(GKG)^l &= G(GKG)^{l-1}(GKG) = G^{2(l-1)}K^{l-1}G(GKG) \\ &= G^{2(l-1)}K^{l-1}G^2KG = G^{2l}K^lG \end{aligned}$$

The last equality in Eq. (A6) follows from Eq. (A4). ■

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